## Exponential Functions

### Exponents Review

Recall:

\[ 3^5 = 3 \times 3 \times 3 \times 3 \times 3 \]

\[ x^5 = x \times x \times x \times x \times x \]

\[ y = a^x \]

### Laws of Exponents

<table>
<thead>
<tr>
<th>Name</th>
<th>Law</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Product</strong></td>
<td>( a^m \cdot a^n = a^{m+n} )</td>
<td>( 3^2 \cdot 3^5 = 3^{2+5} = 3^7 )</td>
</tr>
<tr>
<td><strong>Quotient</strong></td>
<td>( a^m \div a^n = \frac{a^m}{a^n} = a^{m-n} )</td>
<td>( 3^5 \div 3^2 = \frac{3^5}{3^2} = 3^{5-2} = 3^3 )</td>
</tr>
<tr>
<td><strong>Power</strong></td>
<td>( (a^m)^n = a^{mn} )</td>
<td>( (3^2)^5 = 3^{2 \cdot 5} = 3^{10} )</td>
</tr>
<tr>
<td><strong>Negative Exponent</strong></td>
<td>( a^{-m} = \frac{1}{a^m} )</td>
<td>( 3^{-5} = \frac{1}{3^5} )</td>
</tr>
<tr>
<td><strong>Zero Exponent</strong></td>
<td>( a^0 = 1 )</td>
<td>( 3^0 = 1 )</td>
</tr>
<tr>
<td><strong>Fractional Exponent</strong></td>
<td>( \frac{\sqrt[m]{a^m}}{a^n} = \sqrt[m]{a^n} )</td>
<td>( \frac{\sqrt[3]{3^2}}{3^5} = \sqrt[3]{3^{2/5}} )</td>
</tr>
<tr>
<td><strong>Power of a Product</strong></td>
<td>( (ab)^m = a^m \cdot b^m )</td>
<td>( (3 \cdot 4)^5 = 3^5 \cdot 4^5 )</td>
</tr>
<tr>
<td><strong>Power of a Fraction</strong></td>
<td>( \left( \frac{a}{b} \right)^m = \frac{a^m}{b^m} )</td>
<td>( \left( \frac{3}{4} \right)^5 = \frac{3^5}{4^5} )</td>
</tr>
</tbody>
</table>
Solving Exponents

Ex:

a) \(5^3 = x\)  
\(125 = x\)

b) \(2^{-3} = x\)  
\(\frac{1}{8} = x\)

c) \(3^x = \frac{1}{27}\)
\(3^{-3} = \frac{1}{27}\)

So...
\(x = -3\)

d) \(7^{(5-x)} = \frac{1}{49}\)
\(7^2 = 49\)

So...
\(7^{-2} = \frac{1}{49}\)

So...
\(-2 = 5 - x\)
\(-7 = -x\)
\(7 = x\)

e) \(x^6 = 64\)
\(x = \sqrt[6]{64}\)

f) \(5^{2x} = 625\)
\(5^4 = 625\)

So...
\(4 = 2x\)
\(2 = x\)
Simplify Using Laws of Exponents

a) \( \left( \frac{x^5}{x^3} \right)(x^{-1}) \)
\[ = (x^2)(x^{-1}) \]
\[ = x \]

b) \( 6 \left( \frac{a^2}{b^6} \right) \left( \frac{b^{-2}}{a^4} \right) \)
\[ = 6 \left( \frac{a^2 b^{-2}}{b^6 a^4} \right) \]
\[ = 6(a^{-2})(b^{-8}) \]
\[ = \frac{6}{a^2 b^8} \]

c) \( x^7 \left( \frac{x^{-2}}{x^{-3}} \right)^3 \)
\[ = x^7(x^3)^3 \]
\[ = x^16 \]

d) \( (-3a)^4(a^4b^7) \)
\[ = (-3)^4a^4(a^4b^7) \]
\[ = 81a^8b^7 \]

e) \( (4x^2)(xy^3)^2 \)
\[ = \frac{(4x^2)y^6}{2x^4y^6z^{-3}} \]
\[ = 2 \]
\[ = 2z^3 \]

f) \( \sqrt{\frac{25^{-3}}{5^4}} \)
\[ = \left( \frac{25}{5^4} \right)^{-3/2} \]
\[ = (5^{2})^{-3/2} \]
\[ = 5^3 \]
\[ = 125 \]

g) \( \frac{16^3 \cdot 4^2}{4^3} \)
\[ = \frac{(4^2)^3 \cdot 4^2}{4^3} \]
\[ = \frac{4^6 \cdot 4^2}{4^3} \]
\[ = \frac{4^8}{4^3} \]
\[ = 4^5 \]
\[ = 1024 \]

h) \( \frac{3\sqrt{81} \cdot \sqrt{9} \cdot 3}{27} \)
\[ = \frac{3\sqrt{3^4} \cdot \sqrt{3^2} \cdot 3}{3^3} \]
\[ = \frac{3^{4/3} \cdot 3 \cdot 3}{3^3} \]
\[ = \frac{3^{10/3}}{3^3} \]
\[ = 3^{(10/3)-3} \]
\[ = 3^{1/3} \]
Exponential Basic Function

The basic exponential function is \( y = c^x \), where \( c > 0 \) and \( c \neq 1 \)

The base \( c \) determines whether the function is increasing or decreasing
- If \( c > 1 \) the function is increasing
- If \( 0 < c < 1 \) the function is decreasing

The base \( c \) also determines the steepness or the curve
- If \( c > 1 \), a larger \( c \) value leads to a steeper curve
- If \( 0 < c < 1 \), a smaller \( c \) value leads to a steeper curve

The basic function as an asymptote at \( y = 0 \)

Sketching Exponential Functions

To sketch an exponential function, create a table of values, plot the points, and connect the dots, remembering the asymptote.

Ex: Create a table of values and sketch the following functions

a) \( y = 2^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>( \frac{1}{8} )</td>
</tr>
<tr>
<td>-2</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>-1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

b) \( y = \left(\frac{1}{2}\right)^x \) or \( y = 2^{-x} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>8</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{8} )</td>
</tr>
</tbody>
</table>
c) $y = 3^x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>$\frac{1}{27}$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$\frac{1}{9}$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$3$</td>
</tr>
<tr>
<td>$2$</td>
<td>$9$</td>
</tr>
<tr>
<td>$3$</td>
<td>$27$</td>
</tr>
</tbody>
</table>


d) $y = \left(\frac{1}{3}\right)^x$ or $y = 3^{-x}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>$27$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$9$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$3$</td>
</tr>
<tr>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$2$</td>
<td>$\frac{1}{9}$</td>
</tr>
<tr>
<td>$3$</td>
<td>$\frac{1}{27}$</td>
</tr>
</tbody>
</table>
Exponential Transformed Function

Just like the other functions we’ve looked at this year, the transformed exponential function has the parameters $a$, $b$, $h$, and $k$.

$$y = a c^{b(x-h)} + k$$

where $c \neq 1, c > 0, a \neq 0, b \neq 0$

$a$ and $b$ determine the direction of the curve (increasing or decreasing)

- Negative $a$ is a reflection over the x-axis
- Negative $b$ is a reflection over the y-axis

$y = k$ is the asymptote

$h$ is a vertical shift
Sketching Transformed Exponential Functions

To sketch an exponential function, create a table of values, plot the points, and connect the dots, remembering the asymptote.

Ex: Create a table of values and sketch the following functions

a) \[ y = (-1)2^x = -2^x \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-1/8</td>
</tr>
<tr>
<td>-2</td>
<td>-1/4</td>
</tr>
<tr>
<td>-1</td>
<td>-1/2</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-8</td>
</tr>
</tbody>
</table>

b) \[ y = 2^x + 3 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>3.125</td>
</tr>
<tr>
<td>-2</td>
<td>3.25</td>
</tr>
<tr>
<td>-1</td>
<td>3.5</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
</tbody>
</table>

c) \[ y = -2^{-x} + 6 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>5.5</td>
</tr>
<tr>
<td>2</td>
<td>5.75</td>
</tr>
<tr>
<td>3</td>
<td>5.875</td>
</tr>
</tbody>
</table>

d) \[ y = \left(\frac{1}{2}\right)^{-x} \text{ or } y = 2^x \]
Finding the Rule of Exponential Functions

Recall \( y = a c^{b(x-h)} + k \)

Using the laws of exponents, we can re-write this as \( y = ac^x + k \)

Ex: Using the laws of exponents, re-write each function as \( y = ac^x + k \)

\[
a) \quad y = 7 \left( \frac{1}{2} \right)^{-2(x-2)} + 6
\]

\[
y = 7 \left( \frac{1}{2} \right)^{-2(x-2)} + 6
\]

\[
y = 7 \left( \frac{1}{2} \right)^{(x-2)} + 6
\]

\[
y = 7(4)^{(x-2)} + 6
\]

\[
y = 7(4)^{x} \div 4^2 + 6
\]

\[
y = 7 \cdot 4^x \div 16 + 6
\]

\[
y = 7 \cdot 4^x \cdot \frac{1}{16} + 6
\]

\[
y = \frac{7}{16} \cdot 4^x + 6
\]

\[
b) \quad y = 5(3)^{2x+2} + 7
\]

\[
y = 5(3)^{2(x+1)} + 7
\]

\[
y = 5(3^2)^{(x+1)} + 7
\]

\[
y = 5(9)^{(x+1)} + 7
\]

\[
y = 5 \cdot 9^x \cdot 9^{(1)} + 7
\]

\[
y = 5 \cdot 9^x \cdot 9 + 7
\]

\[
y = 45 \cdot 9^x + 7
\]

We will use this simplified version of the rule when finding the rule.

- \( a \) is the initial value
- \( k \) is the value of the asymptote (or constant)
- \( c \) is the base (or rate of change)
  - ex: “triples” means \( c = 3 \); “doubles” means \( c = 2 \); “half” means \( c = 0.5 \)
  - If rate of change is given as a percent:
    - If increasing, \( c = 1 + \frac{\%}{100} \)
    - If decreasing, \( c = 1 - \frac{\%}{100} \)
    - Ex: loses 30% means \( c = 1 - \frac{30}{100} = 0.7 \)
    - Ex: increases 35% means \( c = 1 + \frac{35}{100} = 1.35 \)
    - Ex: earns 4.25% interest means \( c = 1 + \frac{4.25}{100} = 1.0425 \)
    - Ex: depreciates by 12% means \( c = 1 - \frac{12}{100} = 0.88 \)
To find a rule from words, determine a, c, and k and write as a rule

a) You invest $2000 into a savings account with an interest rate of 4.125% compounded annually.

\[ a = 2000 \quad c = 1 + \frac{4.125}{100} = 1.0415 \quad k = 0 \]

\[ y = 2000^{1.0415} \]

b) 200 bacteria are growing in a petri dish. Every seven days the population increases by 17%.

\[ a = 200 \quad c = 1 + \frac{17}{100} = 1.17 \quad k = 0 \]

\[ y = 200^{1.17} \]

c) There are currently 125 fish in a lake. The population of fish is doubling every year.

\[ a = 125 \quad c = 2 \quad k = 0 \]

\[ y = 125^2 \]

d) A ball is dropped from 2m above a table that is 0.75m above the floor. The ball only regains 80% of its height after each bounce.

\[ a = 2 \quad c = 1 - \frac{20}{100} = 0.8 \quad k = 0.75 \]

\[ y = 2^{0.8} + 0.75 \]
To find a rule given 2 points (given in a table of value, on a graph, or in a word problem) we can find the rule using a system of equations

1. Use \( y = ac^x + k \)
2. Plug in points to make 2 equations
3. Eliminate 1 variable by dividing (usually eliminate a)
4. Solve for c
5. Use c to solve for a
6. Write the rule

a) Find the rule of an exponential function (using \( y = ac^x \)) passing through the points (1, 24) and (4, 648)

Step 1

\[ y = ac^x \]

Step 2

Function 1: \( 648 = ac^4 \)
Function 2: \( 24 = ac^1 \)

Step 3

\[ 648 = ac^4 \]
\[ 24 = ac^1 \]
\[ \frac{648}{24} = ac^3 \]
\[ 27 = c^3 \]

Step 4

\[ 27 = c^3 \]
\[ \sqrt[3]{27} = c \]
\[ 3 = c \]

Step 5

\[ 648 = ac^4 \]
\[ 648 = a(3)^4 \]
\[ 648 = 81a \]
\[ 8 = a \]

Step 6

\[ y = 8(3)^x \]
b) Find the rule of an exponential function (using \( y = ac^x \)) passing through the points \((-2, -0.16)\) and \((4, -2500)\)

Step 1

\( y = ac^x \)

Step 2 

Function 1: \(-2500 = ac^4\)
Function 2: \(-0.16 = ac^{-2}\)

Step 3

\(-2500 = ac^4\)  
\(-0.16 = ac^{-2}\)

Step 4 

\(15625 = c^6\)
\(\sqrt[6]{15625} = c\) 
\(5 = c\)

Step 5

\(-0.16 = ac^{-2}\)  
\(-0.16 = a(5)^{-2}\)  
\(-0.16 = 0.04a\)  
\(-4 = a\)

Step 6

\(y = -4(5)^x\)

c) Find the rule of an exponential function passing through the points \((1, 7.4)\) and \((2, 5.78)\) with an asymptote at \(y = 5\)

Step 1

\(y = ac^x + k\)

Step 2 

Function 1: \(7.4 = ac^1 + 5\)  
\(2.4 = ac^1\)  
\(\div 2.4 = ac^1\)
Function 2: \(5.78 = ac^2 + 5\)  
\(0.78 = ac^2\)

Step 3

\(0.78 = ac^2\)  
\(7.4 = ac^1 + 5\)  
\(7.4 = a(0.325)^1 + 5\)  
\(2.4 = 0.325a\)  
\(7.38 = a\)

Step 6

\(y = 7.38(0.325)^x + 5\)
Solving Exponential Functions

To solve for \( y \), plug in \( x \) and solve.

Ex: Solve \( f(x) = 3(4)^{x-2} - 192 \) when \( x = 4 \)

\[
\begin{align*}
    f(x) &= 3(4)^{x-2} - 192 \\
    f(4) &= 3(4)^{4-2} - 192 \\
    f(4) &= 3(4)^2 - 192 \\
    f(4) &= 3 \cdot 16 - 192 \\
    f(4) &= 48 - 192 \\
    f(4) &= -144 \\
\end{align*}
\]

To solve for \( x \)

- Isolate the base(s)
- Make the bases the same
- When the bases are the same, exponents must be equal, so solve for \( x \)
- Note: We will also solve exponentials using logarithmic functions once we learn those

Ex: Solve for \( x \)

\( a) \)

\[
\begin{align*}
    3(4)^{x-2} - 192 &= 0 \\
    3(4)^{x-2} &= 192 \\
    (4)^{x-2} &= 64 \\
    (4)^{x-2} &= 4^3 \\
    x - 2 &= 3 \\
    x &= 5
\end{align*}
\]

\( b) \)

\[
\begin{align*}
    625 \left( \frac{1}{5} \right)^{3x} - 1 &= 0 \\
    625 \left( \frac{1}{5} \right)^{3x} &= 1 \\
    \left( \frac{1}{5} \right)^{3x} &= \frac{1}{625} \\
    \left( \frac{1}{5} \right)^{3x} &= \left( \frac{1}{5} \right)^4 \\
    3x &= 4 \\
    x &= \frac{4}{3}
\end{align*}
\]

\( c) \)

\[
\begin{align*}
    11(7)^{2x-1} &= 539 \\
    (7)^{2x-1} &= 49 \\
    (7)^{2x-1} &= 7^2 \\
    2x - 1 &= 2 \\
    2x &= 3 \\
    x &= 1.5
\end{align*}
\]

\( d) \)

\[
\begin{align*}
    \left( \frac{1}{4} \right)^{8x} &= 2^{-10x+18} \\
    (2^{-2})^{8x} &= 2^{-10x+18} \\
    (2)^{-16x} &= 2^{-10x+18} \\
    -16x &= -10x + 18 \\
    -6x &= 18 \\
    x &= -3
\end{align*}
\]
e) \[ \left( \frac{1}{2} \right)^x + 4 = 0 \]

There is an asymptote at \( y = 4 \) and the function is increasing, so it never touches 0.

No solution.

g) \[
\begin{align*}
(3^{x+4})^2 &= \left( \frac{1}{9} \right)^x \\
3^{2x+8} &= \left( \frac{1}{9} \right)^x \\
3^{2x+8} &= (3^{-2})^x \\
3^{2x+8} &= (3)^{-2x}
\end{align*}
\]

\[2x + 8 = -2x\]
\[4x = -8\]
\[x = -2\]

h) \[
\begin{align*}
27(9)^{x+6} - 6 &= \sqrt{3^{10}} - 6 \\
27(9)^{x+6} &= \sqrt{3^{10}} \\
27(9)^{x+6} &= 243 \\
(9)^{x+6} &= 9
\end{align*}
\]

\[x + 6 = 1\]
\[x = -5\]

i) \[
\begin{align*}
27(9)^x &= \sqrt{3^{10}} \\
27(9)^x &= 3^{\frac{10}{2}} \\
3^3 \cdot (9)^x &= 3^5 \\
3^3 \cdot (3)^{2x} &= 3^5 \\
3^{3+2x} &= 3^5
\end{align*}
\]

\[3 + 2x = 5\]
\[2x = 2\]
\[x = 1\]

1s are your friends!

Anything to the power of 0 is 1.
Solving Exponential Inequalities

To solve exponential inequalities, change inequality to =, solve, and then consult a sketch to answer the question over the appropriate domain (or use a test point).

To use a test point:

- pick a number either larger or smaller than the solution
- Plug that into the inequality and solve
- If you get a true statement, you know the answer must include that point. If you get a false statement, you know the solution cannot include that point.
- All solutions will either be \([-\infty, x]],[x, \infty[ or ]x, \infty[

Ex: Solve \(234(3)^{-0.08x} - 26 > 0\)

\[
\begin{align*}
234(3)^{-0.08x} - 26 & > 0 \\
234(3)^{-0.08x} - 26 & > 0 \\
234(3)^{-0.08x} & = 26 \\
(3)^{-0.08x} & = \frac{1}{9} \\
(3)^{-0.08x} & = 9^{-1} \\
(3)^{-0.08x} & = 3^{-2} \\
-0.08x & = -2 \\
x & = 25
\end{align*}
\]

Test point
Let \(x = 0\)

\[
\begin{align*}
234(3)^{-0.08x} - 26 & > 0 \\
234(3)^0 - 26 & > 0 \\
234(1) - 26 & > 0 \\
208 & > 0
\end{align*}
\]

TRUE
So solution must include \(x = 0\)
\[
\therefore ]-\infty, 25[
\]

Sketch
\[
\begin{array}{c}
0 \\
10 \\
20 \\
30
\end{array}
\]

\[
\begin{array}{c}
-40 \\
-30 \\
-20 \\
-10 \\
0 \\
10 \\
20 \\
30
\end{array}
\]

Ex: Solve \(81^{x-9} \geq 1\)

\[
\begin{align*}
81^{x-9} & = 1 \\
81^{x-9} & = 81^0 \\
x - 9 & = 0 \\
x & = 9
\end{align*}
\]

Test point
Let \(x = 0\)

\[
\begin{align*}
81^{x-9} & \geq 1 \\
81^{-9} & \geq 1 \\
81^{x-9} & \geq 1
\end{align*}
\]

FALSE
So solution must not include \(x = 0\)
\[
\therefore [9, \infty[
\]

Sketch
\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9
\end{array}
\]

\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9
\end{array}
\]
Inverse of Exponential Functions

To find the inverse of exponential functions, we will need logarithmic functions.