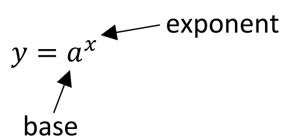
Exponential Functions

Exponents Review

Recall:

$3^5 = 3 \times 3 \times 3 \times 3 \times 3$

$x^5 = x \times x \times x \times x \times x$



Laws of Exponents

Name	Law	Example
Product	$a^m \cdot a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$
Quotient	$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$	$3^5 \div 3^2 = \frac{3^5}{3^2} = 3^{5-2} = 3^3$
Power	$(a^m)^n = a^{m \cdot n}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$
Negative Exponent	$a^{-m} = \frac{1}{a^m}$	$3^{-5} = \frac{1}{3^5}$
Zero Exponent	$a^0 = 1$	$3^0 = 1$
Fractional Exponent	$a^{\frac{m}{n}} = \sqrt[n]{a^m}$	$3^{\frac{2}{5}} = \sqrt[5]{3^2}$
Power of a Product	$(ab)^m = a^m \cdot b^m$	$(3 \cdot 4)^5 = 3^5 \cdot 4^5$
Power of a Fraction	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5}$

Solving Exponents

Ex:

a)

$$5^3 = x$$
$$125 = x$$

b)

$$2^{-3} = x$$
$$\frac{1}{2^3} = x$$
$$\frac{1}{8} = x$$

c)

$$3^x = \frac{1}{27}$$
$$3^3 = 27$$

So…

$$3^{-3} = \frac{1}{27}$$

So…

$$x = -3$$

d)

$$7^{(5-x)} = \frac{1}{49}$$
$$7^2 = 49$$

So…

$$7^{-2} = \frac{1}{49}$$

So…

$$-2 = 5 - x$$
$$-7 = -x$$
$$7 = x$$

e)

$$x^6 = 64$$
$$x = \sqrt[6]{64}$$
$$x = 2$$

f)

$$5^{2x} = 625$$
$$5^4 = 625$$

So…

$$4 = 2x$$
$$2 = x$$

Simplify Using Laws of Exponents

a)

$$\left(\frac{x^5}{x^3}\right)(x^{-1})$$
$$= (x^2)(x^{-1})$$
$$= x$$

b)

$$6\left(\frac{a^2}{b^6}\right)\left(\frac{b^{-2}}{a^4}\right)$$
$$= 6\left(\frac{a^2b^{-2}}{b^6a^4}\right)$$
$$= 6(a^{-2})(b^{-8})$$
$$= \frac{6}{a^2b^8}$$

c)

$$x^7\left(\frac{x^{-2}}{x^{-5}}\right)^3$$
$$= x^7(x^3)^3$$
$$= x^7(x)^9$$
$$= x^{16}$$

d)

$$(-3a)^4(a^4b^7)$$
$$= (-3)^4a^4(a^4b^7)$$
$$= 81a^8b^7$$

e)

$$\frac{(4x^2)(xy^3)^2}{2x^4y^6z^{-3}}$$
$$= \frac{(4x^2)x^2y^6}{2x^4y^6z^{-3}}$$
$$= \frac{4x^4y^6}{2x^4y^6z^{-3}}$$
$$= \frac{2}{z^{-3}}$$
$$= 2z^3$$

f)

$$\sqrt{\left(\frac{25}{5^4}\right)^{-3}}$$
$$= \left(\frac{25}{5^4}\right)^{-3/2}$$
$$= \left(\frac{5^2}{5^4}\right)^{-3/2}$$
$$= (5^{-2})^{-3/2}$$
$$= 5^3$$
$$= 125$$

g)

$$\frac{16^3 \cdot 4^2}{4^3}$$
$$= \frac{(4^2)^3 \cdot 4^2}{4^3}$$
$$= \frac{4^6 \cdot 4^2}{4^3}$$
$$= \frac{4^8}{4^3}$$
$$= 4^5$$
$$= 1024$$

h)

$$\frac{\sqrt[3]{81} \cdot \sqrt{9} \cdot 3}{27}$$
$$= \frac{\sqrt[3]{3^4} \cdot \sqrt{3^2} \cdot 3}{3^3}$$
$$= \frac{3^{4/3} \cdot 3 \cdot 3}{3^3}$$
$$= \frac{3^{10/3}}{3^3}$$
$$= 3^{\left(\frac{10}{3}-3\right)}$$
$$= 3^{1/3}$$

Exponential Basic Function

The basic exponential function is $y = c^x$, where $c > 0 \ and \ c \neq 1$

The base (c) determines whether the function is increasing or decreasing

- If $c > 1$ the function is increasing
- If $0 < c < 1$ the function is decreasing

The base (c) also determines the steepness or the curve

- If $c > 1$, a larger c value leads to a steeper curve
- If $0 < c < 1$, a smaller c value leads to a steeper curve

The basic function as an asymptote at $y = 0$

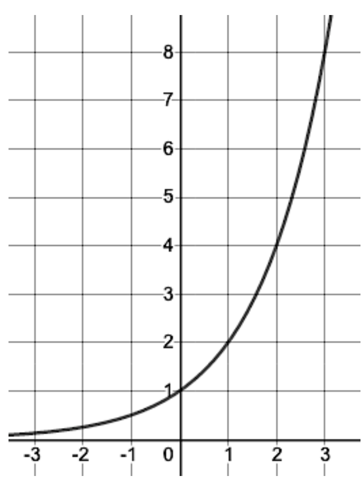
Sketching Exponential Functions

To sketch an exponential function, create a table of values, plot the points, and connect the dots, remembering the asymptote.

Ex: Create a table of values and sketch the following functions

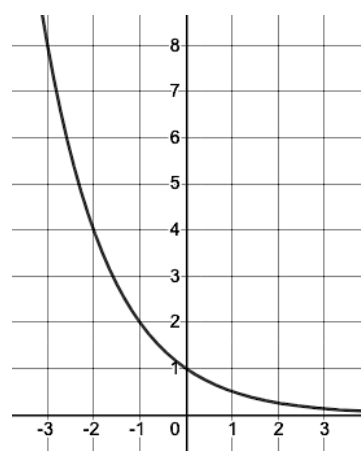
a) $y = 2^x$

x	**y**
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



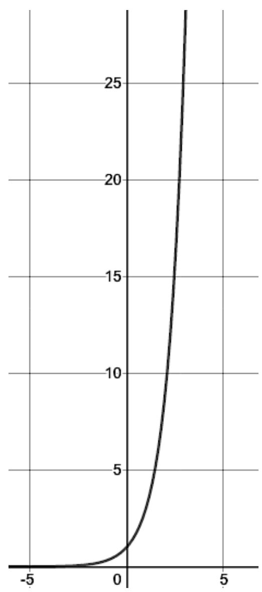
b) $y = \left(\frac{1}{2}\right)^x \ or \ y = 2^{-x}$

x	y
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



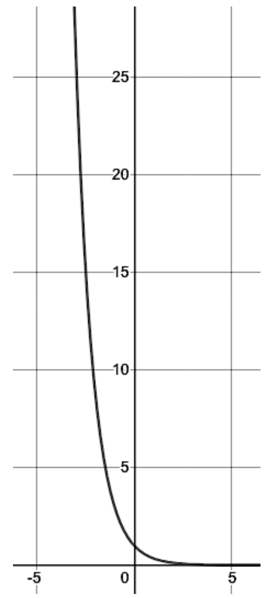
c) $y = 3^x$

x	**y**
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27



d) $y = \left(\frac{1}{3}\right)^x \; or \; y = 3^{-x}$

x	y
-3	27
-2	9
-1	3
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$
3	$\frac{1}{27}$



Exponential Transformed Function

Just like the other functions we've looked at this year, the transformed exponential function has the parameters a, b, h, and k.

$$y = ac^{b(x-h)} + k$$

where $c \neq 1, c > 0, a \neq 0, b \neq 0$

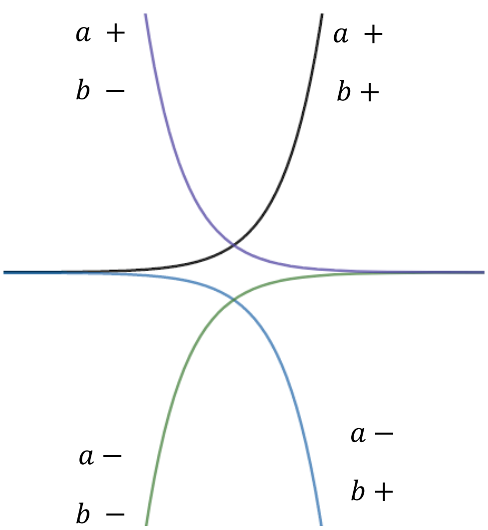
a and b determine the direction of the curve (increasing or decreasing)

- Negative a is a reflection over the x-axis
- Negative b is a reflection over the y-axis

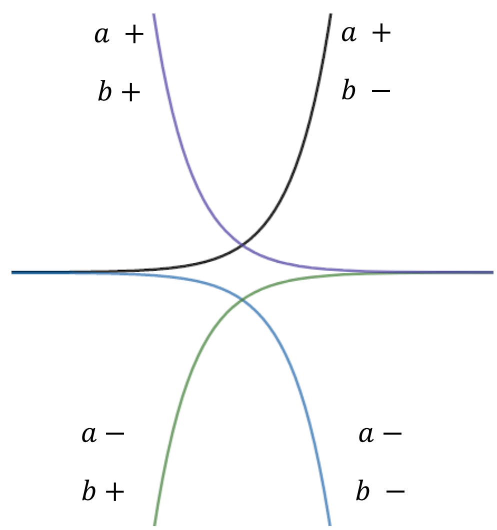
$y = k$ is the asymptote

h is a vertical shift

<u>If $c > 1$</u>



<u>If $0 < c < 1$</u>



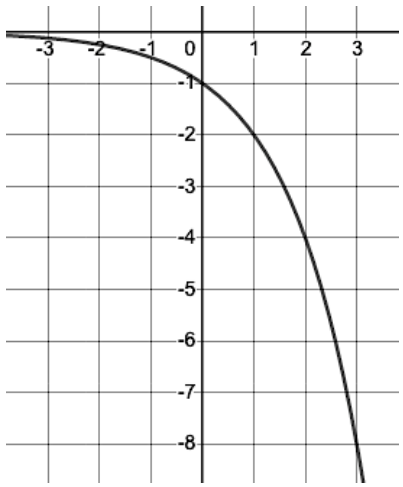
Sketching Transformed Exponential Functions

To sketch an exponential function, create a table of values, plot the points, and connect the dots, remembering the asymptote.

Ex: Create a table of values and sketch the following functions

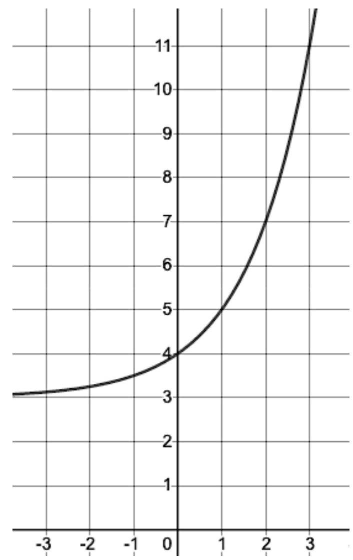
a) $y = (-1)2^x = -2^x$

x	y
-3	$-\frac{1}{8}$
-2	$-\frac{1}{4}$
-1	$-\frac{1}{2}$
0	-1
1	-2
2	-4
3	-8



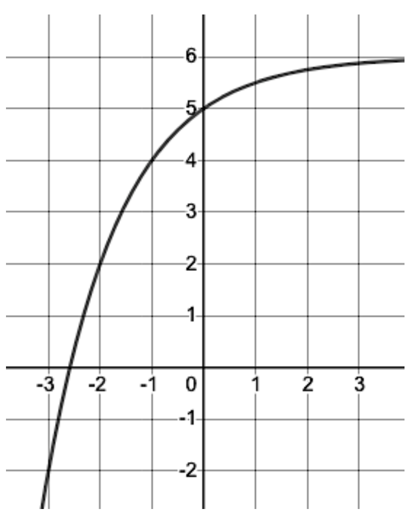
b) $y = 2^x + 3$

x	y
-3	3.125
-2	3.25
-1	3.5
0	4
1	5
2	9
3	11



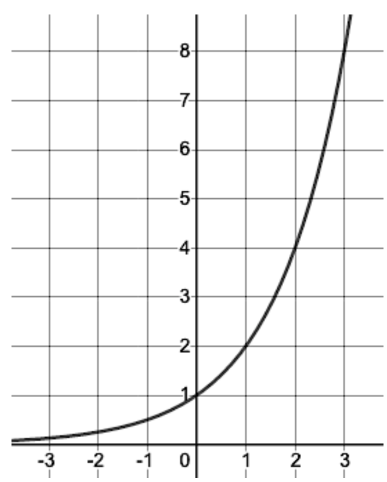
c) $y = -2^{-x} + 6$

x	y
-3	-2
-2	2
-1	4
0	5
1	5.5
2	5.75
3	5.875



d) $y = \left(\frac{1}{2}\right)^{-x}$ $or\ y = 2^x$

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Finding the Rule of Exponential Functions

Recall $y = ac^{b(x-h)} + k$

Using the laws of exponents, we can re-write this as $y = ac^{x} + k$

Ex: Using the laws of exponents, re-write each function as $y = ac^{x} + k$

a) $y = 7\left(\frac{1}{2}\right)^{-2(x-2)} + 6$

$$y = 7\left(\frac{1}{2}\right)^{-2(x-2)} + 6$$

$$y = 7\left(\left(\frac{1}{2}\right)^{-2}\right)^{(x-2)} + 6$$

$$y = 7(4)^{(x-2)} + 6$$

$$y = 7(4)^{(x)} \div 4^{2} + 6$$

$$y = 7 \cdot 4^{x} \div 16 + 6$$

$$y = 7 \cdot 4^{x} \cdot \frac{1}{16} + 6$$

$$y = \frac{7}{16} \cdot 4^{x} + 6$$

b) $y = 5(3)^{2x+2} + 7$

$$y = 5(3)^{2x+2} + 7$$

$$y = 5(3)^{2(x+1)} + 7$$

$$y = 5(3^{2})^{(x+1)} + 7$$

$$y = 5(9)^{(x+1)} + 7$$

$$y = 5 \cdot 9^{x} \cdot 9^{(1)} + 7$$

$$= 5 \cdot 9^{x} \cdot 9 + 7$$

$$= 45 \cdot 9^{x} + 7$$

We will use this simplified version of the rule when finding the rule.

- a is the initial value
- k is the value of the asymptote (or constant)
- c is the base (or rate of change)
    - ex: "triples" means $c = 3$; "doubles" means $c = 2$; "half" means $c = 0.5$
    - If rate of change is given as a percent:
        - If increasing, $c = 1 + \frac{\%}{100}$
        - If decreasing, $c = 1 - \frac{\%}{100}$
        - Ex: loses 30% means $c = 1 - \frac{30}{100} = 0.7$
        - Ex: increases 35% means $c = 1 + \frac{35}{100} = 1.35$
        - Ex: earns 4.25% interest means $c = 1 + \frac{4.25}{100} = 1.0425$
        - Ex: depreciates by 12% means $c = 1 - \frac{12}{100} = 0.88$

To find a rule from words, determine a, c, and k and write as a rule

a) You invest $2000 into a savings account with an interest rate of 4.125% compounded annually.

$a = 2000$ $\qquad c = 1 + \frac{4.125}{100} = 1.0415$ $\qquad k = 0$

$$y = 2000^{1.0415}$$

b) 200 bacteria are growing in a petri dish. Every seven days the population increases by 17%.

$a = 200$ $\qquad c = 1 + \frac{17}{100} = 1.17$ $\qquad k = 0$

$$y = 200^{1.17}$$

c) There are currently 125 fish in a lake. The population of fish is doubling every year.

$a = 125$ $\qquad c = 2$ $\qquad k = 0$

$$y = 125^{2}$$

d) A ball is dropped from 2m above a table that is 0.75m above the floor. The ball only regains 80% of its height after each bounce.

$a = 2$ $\qquad c = 1 - \frac{20}{100} = 0.8$ $\qquad k = 0.75$

$$y = 2^{0.8} + 0.75$$

To find a rule given 2 points (given in a table of value, on a graph, or in a word problem) we can find the rule using a system of equations

1. Use $y = ac^x + k$
2. Plug in points to make 2 equations
3. Eliminate 1 variable by dividing (usually eliminate a)
4. Solve for c
5. Use c to solve for a
6. Write the rule

a) Find the rule of an exponential function (using $y = ac^x$) passing through the points

(1, 24) and (4, 648)

Step 1

$$y = ac^x$$

Step 2

Function 1: $648 = ac^4$

Function 2: $24 = ac^1$

Step 3

$$648 = ac^4$$

$$\underline{\div\, 24 = ac^1}$$

$$27 = c^3$$

Step 4

$$27 = c^3$$

$$\sqrt[3]{27} = c$$

$$3 = c$$

Step 5

$$648 = ac^4$$

$$648 = a(3)^4$$

$$648 = 81a$$

$$8 = a$$

Step 6

$$y = 8(3)^x$$

b) Find the rule of an exponential function (using $y = ac^x$) passing through the points $(-2, -0.16)$ and $(4, -2500)$

Step 1

$$y = ac^x$$

Step 2

Function 1: $-2500 = ac^4$

Function 2: $-0.16 = ac^{-2}$

Step 3

$$-2500 = ac^4$$

$$\underline{\div -0.16 = ac^{-2}}$$

$$15625 = c^6$$

Step 4

$$15625 = c^6$$

$$\sqrt[6]{15625} = c$$

$$5 = c$$

Step 5

$$-0.16 = ac^{-2}$$

$$-0.16 = a(5)^{-2}$$

$$-0.16 = 0.04a$$

$$-4 = a$$

Step 6

$$y = -4(5)^x$$

c) Find the rule of an exponential function passing through the points $(1, 7.4)$ and $(2, 5.78)$ with an asymptote at $y = 5$

Step 1

$$y = ac^x + k$$

Step 2

Function 1: $7.4 = ac^1 + 5$

$$2.4 = ac^1$$

Function 2: $5.78 = ac^2 + 5$

$$0.78 = ac^2$$

Step 3

$$0.78 = ac^2$$

$$\underline{\div 2.4 = ac^1}$$

$$0.325 = c$$

Step 4

$$0.325 = c$$

Step 5

$$7.4 = ac^1 + 5$$

$$7.4 = a(0.325)^1 + 5$$

$$2.4 = 0.325a$$

$$7.38 = a$$

Step 6

$$y = 7.38(0.325)^x + 5$$

Solving Exponential Functions

To solve for y, plug in x and solve.

Ex: Solve $f(x) = 3(4)^{x-2} - 192$ when $x = 4$

$$f(x) = 3(4)^{x-2} - 192$$
$$f(4) = 3(4)^{4-2} - 192$$
$$f(4) = 3(4)^2 - 192$$
$$f(4) = 3 \cdot 16 - 192$$
$$f(4) = 48 - 192$$
$$f(4) = -144$$

To solve for x

- Isolate the base(s)
- Make the bases the same
- When the bases are the same, exponents must be equation, so solve for x
- Note: We will also solve exponentials using logarithmic functions once we learn those

Ex: Solve for x

a)

$$3(4)^{x-2} - 192 = 0$$
$$3(4)^{x-2} = 192$$
$$(4)^{x-2} = 64$$
$$(4)^{x-2} = 4^3$$

$$x - 2 = 3$$
$$x = 5$$

b)

$$625\left(\frac{1}{5}\right)^{3x} - 1 = 0$$
$$625\left(\frac{1}{5}\right)^{3x} = 1$$
$$\left(\frac{1}{5}\right)^{3x} = \frac{1}{625}$$
$$\left(\frac{1}{5}\right)^{3x} = \frac{1}{5^4}$$
$$\left(\frac{1}{5}\right)^{3x} = \left(\frac{1}{5}\right)^4$$

$$3x = 4$$
$$x = \frac{4}{3}$$

c)

$$11(7)^{2x-1} = 539$$
$$(7)^{2x-1} = 49$$
$$(7)^{2x-1} = 7^2$$

$$2x - 1 = 2$$
$$2x = 3$$
$$x = 1.5$$

d)

Point of intersection

$$\left(\frac{1}{4}\right)^{8x} = 2^{-10x+18}$$
$$(2^{-2})^{8x} = 2^{-10x+18}$$
$$(2)^{-16x} = 2^{-10x+18}$$

$$-16x = -10x + 18$$
$$-6x = 18$$
$$x = -3$$

e)

$$\left(\frac{1}{2}\right)^x + 4 = 0$$

There is an asymptote at $y = 4$ and the function is increasing, so it never touches 0.

No solution.

f)

$$27(9)^{x+6} - 6 = \sqrt{3^{10}} - 6$$
$$27(9)^{x+6} = \sqrt{3^{10}}$$
$$27(9)^{x+6} = 243$$
$$(9)^{x+6} = 9$$

$$x + 6 = 1$$
$$x = -5$$

g)

$$(3^{x+4})^2 = \left(\frac{1}{9}\right)^x$$
$$3^{2x+8} = \left(\frac{1}{9}\right)^x$$
$$3^{2x+8} = (3^{-2})^x$$
$$3^{2x+8} = (3)^{-2x}$$

$$2x + 8 = -2x$$
$$4x = -8$$
$$x = -2$$

h)

$$2(3)^{x-4} = 2$$
$$(3)^{x-4} = 1$$
$$(3)^{x-4} = 3^0$$

1s are your friends!

Anything to the power of 0 is 1.

$$x - 4 = 0$$
$$x = 4$$

i)

$$27(9)^x = \sqrt{3^{10}}$$
$$27(9)^x = 3^{\frac{10}{2}}$$
$$3^3 \cdot (9)^x = 3^5$$
$$3^3 \cdot (3)^{2x} = 3^5$$
$$3^{3+2x} = 3^5$$

$$3 + 2x = 5$$
$$2x = 2$$
$$x = 1$$

Solving Exponential Inequalities

To solve exponential inequalities, change inequality to =, solve, and then consult a sketch to answer the question over the appropriate domain (or use a test point).

To use a test point:

- pick a number either larger or smaller than the solution
- Plug that into the inequality and solve
- If you get a true statement, you know the answer must include that point. If you get a false statement, you know the solution cannot include that point.
- All solutions will either be $]-\infty, x],]-\infty, x[, [x, \infty[, or\ ]x, \infty[$

Ex: Solve $234(3)^{-0.08x} - 26 > 0$

$$234(3)^{-0.08x} - 26 > 0$$
$$234(3)^{-0.08x} - 26 = 0$$
$$234(3)^{-0.08x} = 26$$
$$(3)^{-0.08x} = \frac{1}{9}$$
$$(3)^{-0.08x} = 9^{-1}$$
$$(3)^{-0.08x} = 3^{-2}$$

$$-0.08x = -2$$
$$x = 25$$

Test point

Let $x = 0$

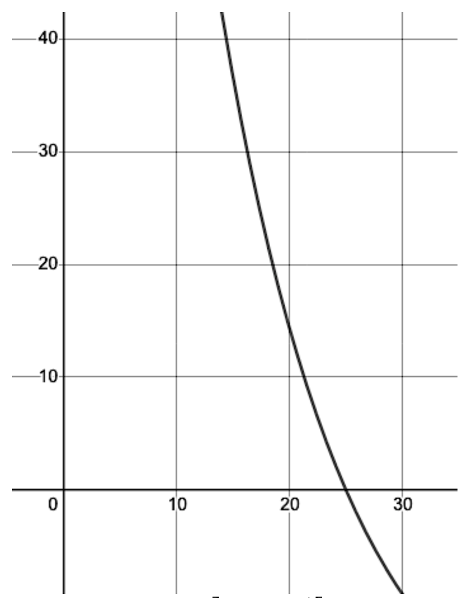
$$234(3)^{-0.08x} - 26 > 0$$
$$234(3)^{0} - 26 > 0$$
$$234(1) - 26 > 0$$
$$234 - 26 > 0$$
$$208 > 0$$

TRUE

So solution must include $x = 0$

$\therefore\ ]-\infty, 25[$

Sketch



$\therefore\ ]-\infty, 25[$

Ex: Solve $81^{x-9} \geq 1$

$$81^{x-9} = 1$$
$$81^{x-9} = 81^{0}$$

$$x - 9 = 0$$
$$x = 9$$

Test point

Let $x = 0$

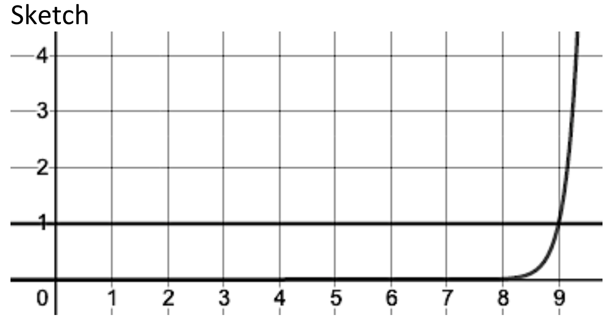
$$81^{x-9} \geq 1$$
$$81^{-9} \geq 1$$
$$81^{x-9} \geq 1$$

FALSE

So solution must not include $x = 0$

$\therefore\ [9, \infty[$

Sketch



$\therefore\ [9, \infty[$

Inverse of Exponential Functions

To find the inverse of exponential functions, we will need logarithmic functions.