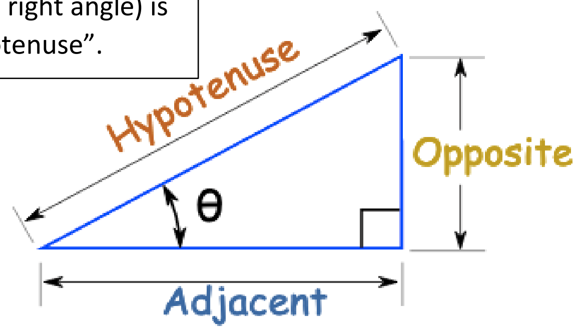
Trigonometry

Trig Ratios

The longest side (and the one across from the right angle) is the “Hypotenuse”.

θ (theta) is the Greek letter used to represent an unknown (reference) angle.



The side across from the reference angle is the “Opposite” side.

The side next to the reference angle is the “Adjacent” side.

Sine (sin), cosine (cos), and tangent (tan) are the three **basic trigonometric functions.**

$$\sin\theta = \frac{o}{h} \qquad \cos\theta = \frac{a}{h} \qquad \tan\theta = \frac{o}{a}$$

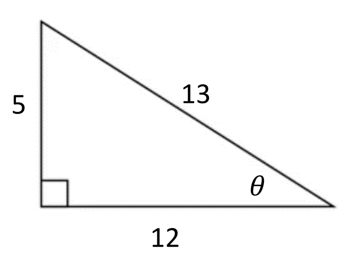
We can also take the reciprocal of sin, cos, and tan, which produces 3 more ratios

$$Cosecant\colon cosec\ \theta\ or\ \csc\theta = \frac{1}{\sin\theta} = \frac{h}{o}$$

$$Secant\colon\ \sec\theta = \frac{1}{\cos\theta} = \frac{h}{a}$$

$$Cotangent\colon \cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

Ex: Find all 6 trig ratios for the angle θ in the right triangle below.



$$\sin\theta = \frac{5}{13} \qquad \cos\theta = \frac{12}{13} \qquad \tan\theta = \frac{5}{12}$$

$$\csc\theta = \frac{13}{5} \qquad \sec\theta = \frac{13}{12} \qquad \cot\theta = \frac{12}{5}$$

Ex: Solve for x

a) $\sin 57° = x$

$$0.8387 = x$$

b) $\sec 30° = \frac{x}{2}$

$$\frac{1}{\cos 30°} = \frac{x}{2}$$

$$\frac{2}{\cos 30°} = x$$

$$2.3094 = x$$

c) $\csc x = 1.4$

$$\frac{1}{\sin x} = \frac{1.4}{1}$$

$$\sin x = \frac{1}{1.4}$$

$$sin^{-1}\left(\frac{1}{1.4}\right) = x$$

$$x = 45.58°$$

d) $\sin 70° = \frac{x}{2}$

$$2 \sin 70 = x$$

$$1.8794 = x$$

e) $\cot 0.7° = x$

$$\frac{1}{\tan 0.7} = x$$

$$81.8470 = x$$

f) $\tan x = 0.25$

$$tan^{-1}(0.25) = x$$

$$14.04° = x$$

g) $\frac{\cos x}{\sin x} = 3$

$$\frac{cos}{sin} = \frac{\frac{a}{h}}{\frac{o}{h}} = \frac{a}{h} \times \frac{h}{o} = \frac{a}{o}$$

$$cot = \frac{a}{o}$$

$$\cot x = 3$$

$$\frac{1}{\tan x} = 3$$

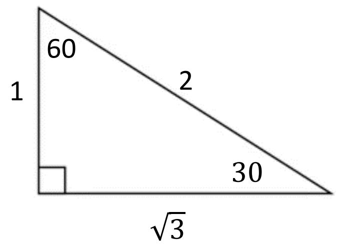
$$\tan x = \frac{1}{3}$$

$$tan^{-1}\left(\frac{1}{3}\right) = x$$

$$18.43° = x$$

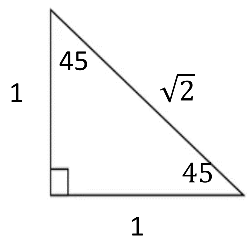
Special Triangles

There are some special triangles (and their trig ratios) that will be important as we move through this unit.



$\sin 30° = \frac{1}{2}$ $\qquad \cos 30° = \frac{\sqrt{3}}{2}$ $\qquad \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\qquad \cos 60° = \frac{1}{2}$ $\qquad \tan 60° = \sqrt{3}$



$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\qquad \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\qquad \tan 45° = \frac{1}{1} = 1$

Other important values:

$\cos 0° = 1$ $\qquad \sin 0° = 0$ $\qquad \tan 0° = 0$

$\cos 90° = 0$ $\qquad \sin 90° = 1$ $\qquad \tan 90° = \textit{does not exist}$

Ex: Find the exact value.

a) $(\sin 45°)(\sin 60°)$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{6}}{2}$

b) $\sec 30°$

$= \frac{1}{\cos 30}$

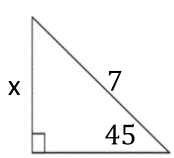
$= \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

c) $\csc 30°$

$= \frac{1}{\sin 30}$

$= \frac{1}{1/2} = 2$

d)

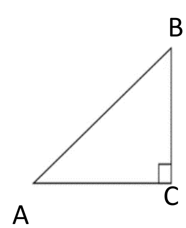


$\sin 45 = \frac{x}{7}$

$\frac{\sqrt{2}}{2} = \frac{x}{7}$

$\frac{7\sqrt{2}}{2} = x$

e) Given $\cos B = \frac{1}{2}$ $\textit{determine}$ $\cot B$



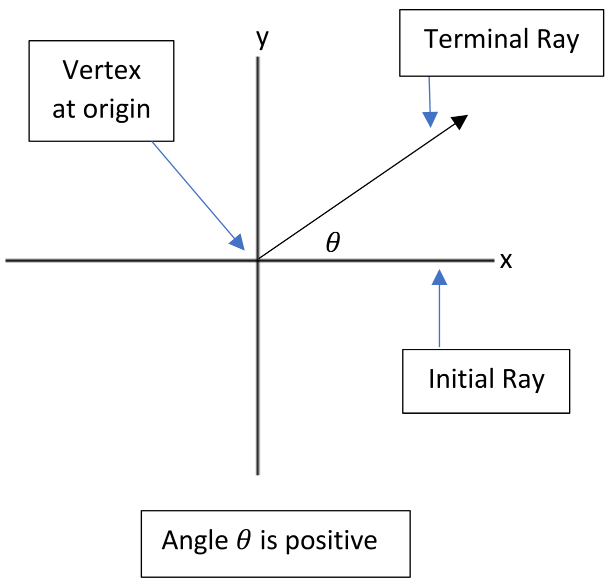
$b = \sqrt{2^2 - 1^2} = \sqrt{3}$

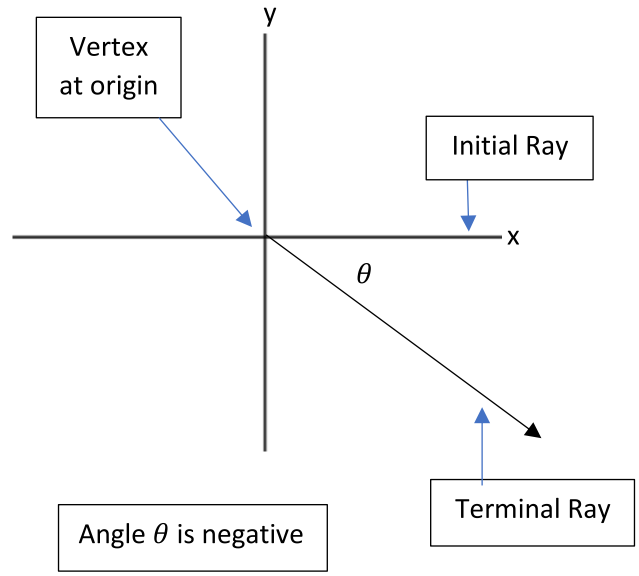
$cot = \frac{a}{o}$

$\cot B = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Trig Ratios and Angles

A trig angle can be seen as an angle of rotation.



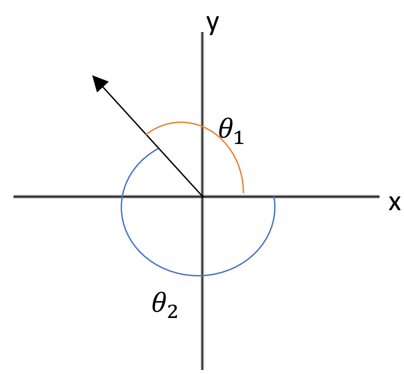


θ is always formed by the x-axis and the terminal ray.

Two angle that share the same terminal ray are co-terminal angles.

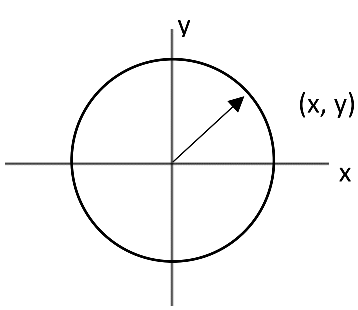
Their positive values add to 360°.

Remember, though that the value of θ_2 is negative.

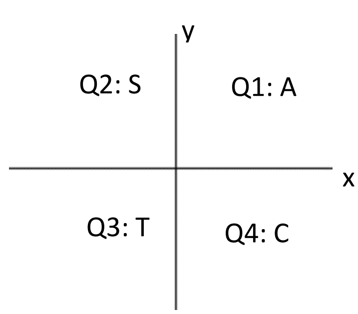


Rotating a terminal ray produces a circule with radius equal to the length of the terminal ray.

The end of the terminal ray can be expressed as an ordered pair. $P(\theta) = (x, y)$



The location of $P(\theta)$ determines the signs of x and y, and therefore also the signs of the trig ratios (CAST Rule)



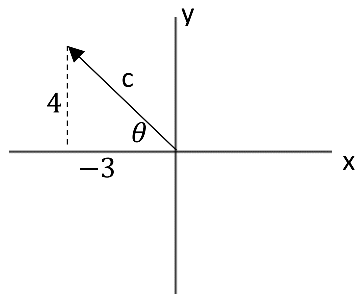
Quadrant 4: **C**osine ratios are positive

Quadrant 1: **A**ll ratios are positive

Quadrant 2: **S**ine ratios are positive

Quadrant 3: **T**an ratios are positive

Ex: Given $P(\theta) = (-3, 4)$ find the value of $\sin\theta$, $\cos\theta$, and $\tan\theta$.



$$a^2 + b^2 = c^2$$

$$(-3)^2 + 4^2 = c^2$$

$$9 + 16 = c^2$$

$$25 = c^2$$

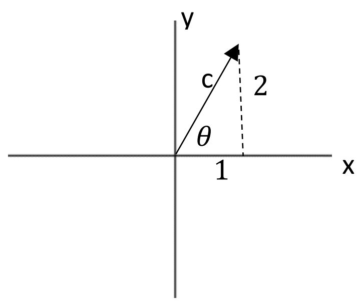
$$5 = c$$

$$\sin\theta = \frac{4}{5}$$

$$\cos\theta = -\frac{3}{5}$$

$$\tan\theta = -\frac{4}{3}$$

Ex: Given $P(\theta) = (1, 2)$ find the value of $\sin\theta$, $\cos\theta$, and $\tan\theta$.



$$a^2 + b^2 = c^2$$

$$(1)^2 + (2)^2 = c^2$$

$$1 + 4 = c^2$$

$$5 = c^2$$

$$\sqrt{5} = c$$

$$\sin\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

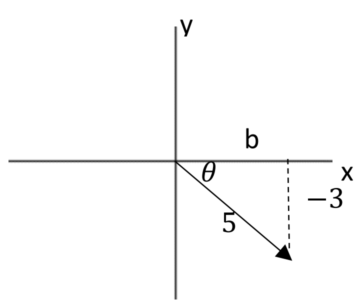
$$\cos\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\tan\theta = \frac{2}{1} = 2$$

Ex: Given $\sin\theta = \frac{-3}{5}$ and $\tan\theta < 0$, find the value of $\cos\theta$

Because $\sin\theta$ is negative, we must be in either Q3 or Q4.

Because $\tan\theta$ is negative, we must be in Q4.



$$(-3)^2 + b^2 = (5)^2$$

$$9 + (2)^2 = 25$$

$$b^2 = 16$$

$$b = 4$$

$$\cos\theta = \frac{4}{5}$$

Solving for θ

1) Solve for reference angle θ_R
2) Find other angles using CAST
3) Check answers against any restrictions listed in the question

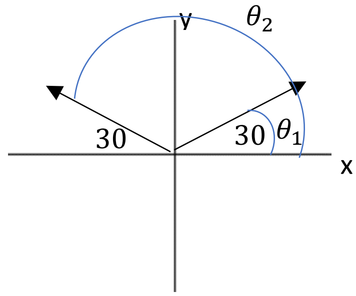
Ex: Given $\sin\theta = 0.5$, find the values for θ over $0° \leq \theta \leq 360°$

$\sin\theta_R = 0.5$

$\theta_R = sin^{-1}(0.5)$

$\theta_R = 30$

Sine is positive, so we are in Q1 and Q2.



$\theta_1 = 30°$

$\theta_2 = 180 - 30 = 150°$

Both fit within the restrictions of the question.

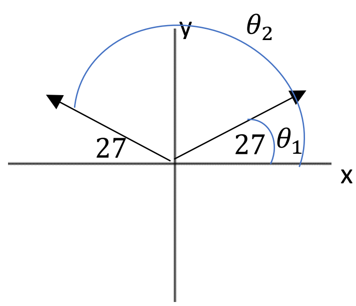
Ex: Given that $\sin\theta = 0.4540$, find the values for θ over $0° \leq \theta \leq 360°$

$\sin\theta_R = 0.4540$

$\theta_R = sin^{-1}(0.4540)$

$\theta_R = 27$

Sine is positive, so we are in Q1 and Q2.



$\theta_1 = 27°$

$\theta_2 = 180 - 27 = 153°$

Both fit within the restrictions of the question.

Ex: Given that $\tan\theta = -0.6249$, find the values for θ over $0° \leq \theta \leq 360°$

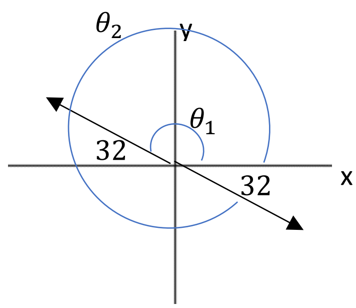
$\tan\theta_R = -0.6249$

$\theta_R = tan^{-1}(0.6249)$

$\theta_R = 32$

If given a negative value for the trig ratio, solve for the reference angle using the positive value.

Tangent is negative, so we are in Q2 and Q4.



$\theta_1 = 180 - 32 = 148°$

$\theta_2 = 360 - 32 = 328°$

Both fit within the restrictions of the question.

Note: if the above example had the restriction: $360° \leq \theta \leq 720°$, what would the answer be?

We have to go all the way around once, then add θ_1 *and* θ_2.

$\theta_1 = 360 + 148 = 508°$

$\theta_2 = 360 + 328 = 688°$

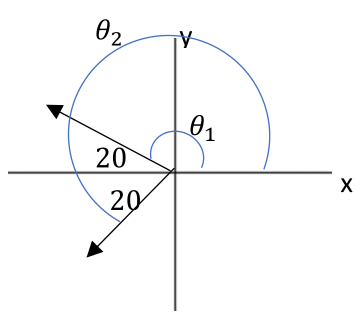
Ex: Given $\cos\theta = -0.9397$ find θ over $180° \leq \theta \leq 270°$

$\cos\theta_R = -0.9397$

$\theta_R = cos^{-1}(0.9397)$

$\theta_R = 20°$

Cosine is negative, so we are in Q2 and Q3.



~~$\theta_1 = 180 - 20 = 160°$~~

$\theta_2 = 180 + 20 = 200°$

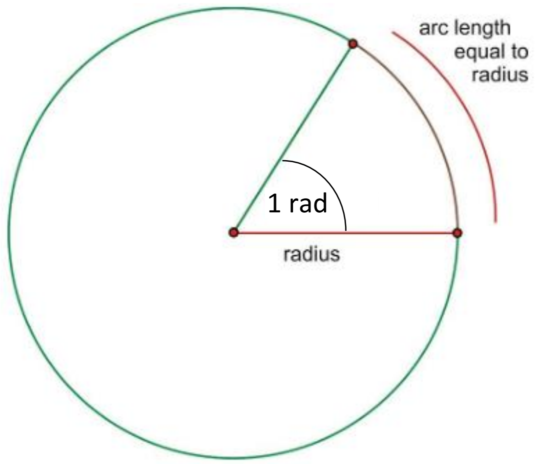
160° does not fit within the restrictions.

Radians and Arc Length

A radian is another way of measuring angles.

A radian is the measure of a central angle whereby the rays of radius r intersect the arc with length r.

When the arc length is the same as the radius, the angle equals 1 rad or 57.30°.



Key angles

$$360° = 2\pi\ radians$$

$$180° = \pi\ radians$$

$$57.30° = 1\ radians$$

Converting between radians and degrees

$$\frac{n°}{180°} = \frac{\theta\ radians}{\pi\ radians}$$

Ex: Convert

a) 45° to radians

$$\frac{45°}{180°} = \frac{\theta\ radians}{\pi\ radians}$$

$$45 \cdot \pi = 180 \cdot \theta$$

$$\frac{45\pi}{180} = \theta$$

$$\frac{\pi}{4} radians = \theta$$

b) 60° to radians

$$\frac{60°}{180°} = \frac{\theta\ radians}{\pi\ radians}$$

$$60 \cdot \pi = 180 \cdot \theta$$

$$\frac{60\pi}{180} = \theta$$

$$\frac{\pi}{3} radians = \theta$$

c) 40° to radians

$$\frac{40°}{180°} = \frac{\theta\ radians}{\pi\ radians}$$

$$40 \cdot \pi = 180 \cdot \theta$$

$$\frac{40\pi}{180} = \theta$$

$$\frac{2\pi}{9} radians = \theta$$

Ex: Convert

a) $\frac{\pi}{2}$ radians to degrees

$$\frac{n°}{180°} = \frac{\left(\frac{\pi}{2}\right)\ radians}{\pi\ radians}$$

$$\frac{n}{180} = \frac{\pi}{2} \cdot \frac{1}{\pi}$$

$$\frac{n}{180} = \frac{1}{2}$$

$$n = 90°$$

b) $\frac{5\pi}{2}$ radians to degrees

$$\frac{n°}{180°} = \frac{\left(\frac{5\pi}{2}\right)\ radians}{\pi\ radians}$$

$$\frac{n}{180} = \frac{5\pi}{2} \cdot \frac{1}{\pi}$$

$$\frac{n}{180} = \frac{5}{2}$$

$$n = 450°$$

c) $\frac{7\pi}{6}$ radians to degrees

$$\frac{n°}{180°} = \frac{\left(\frac{7\pi}{6}\right)\ radians}{\pi\ radians}$$

$$\frac{n}{180} = \frac{7\pi}{6} \cdot \frac{1}{\pi}$$

$$\frac{n}{180} = \frac{7}{6}$$

$$n = 210°$$

Solving for θ in Radians

Use the same steps as above, but restrictions (and final answers) will be given in radians.

You can either solve using degrees and convert to radians, or solve using radians (you must change your calculator to RAD mode).

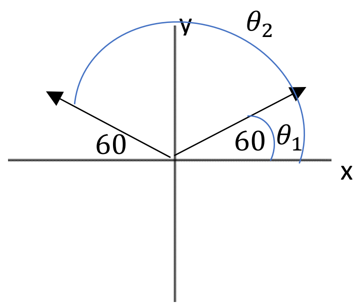
MAKE SURE YOUR CALCULATOR IS IN RAD MODE, NOT DEG

Ex: Given $\sin\theta = \frac{\sqrt{3}}{2}$, find θ over $0 \leq \theta \leq 2\pi$

$$\sin\theta_R = \frac{\sqrt{3}}{2}$$

$$\theta_R = 60$$

Sine is positive, so we are in Q1 and Q2.



$$\theta_1 = 60$$

$$\frac{60°}{180°} = \frac{\theta\ radians}{\pi\ radians}$$

$$\frac{\pi}{3} radians = \theta_1$$

$$\theta_2 = 180 - 60 = 120°$$

$$\frac{120°}{180°} = \frac{\theta\ radians}{\pi\ radians}$$

$$\frac{2\pi}{3} radians = \theta_2$$

Both fit within the restrictions of the question.

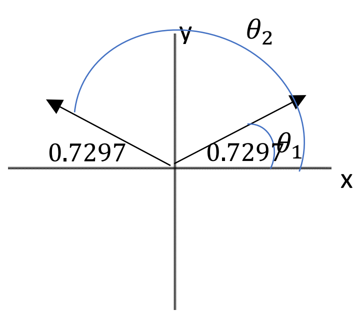
Ex: Given $\sin\theta = \frac{2}{3}$, find θ over $0 \leq \theta \leq 2\pi$

$$\sin\theta_R = \frac{2}{3}$$

$$\theta_R = sin^{-1}\left(\frac{2}{3}\right)$$

$$\theta_R = 0.7297$$

Sine is positive, so we are in Q1 and Q2.



$$\theta_1 = 0.7297\ rad$$

$$\theta_2 = \pi - 0.7297$$

$$\theta_2 = 2.4119\ rad$$

Both fit within the restrictions of the question.

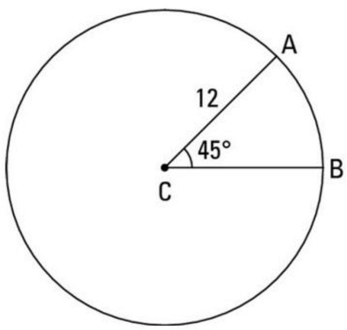
Arc Length

In addition to finding angles, we can find the arc length using either degrees or radians.

Using degrees: $\frac{\theta}{360} = \frac{arc\ length}{2\pi r}$

Using radians: $arc\ length = \theta r$

Ex: Find the arc length of AB

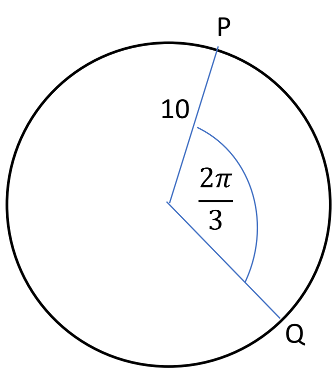


$$\frac{\theta}{360} = \frac{arc\ length}{2\pi r}$$

$$\frac{45}{360} = \frac{arc\ length}{2\pi \cdot 12}$$

$$arc\ length = 9.4248\ units$$

Ex: Find the arc length of PQ



$$arc\ length = \theta r$$

$$arc\ length = \frac{2\pi}{3} \cdot 10$$

$$arc\ length = \frac{20\pi}{3}\ units$$

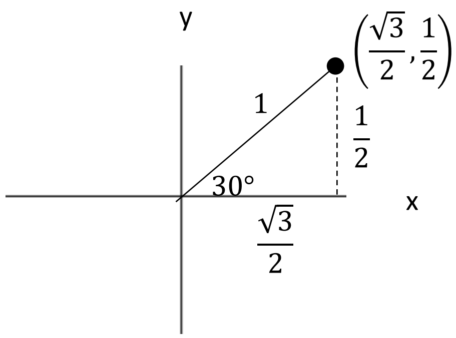
The Unit Circle

The unit circle is a circle centered at the origin of the Cartesian plane with a terminal ray (and therefore a radius) of 1 unit.

Determining the coordinates of the unit circle combines angles (in degrees and radians), trig points, special triangles, and trig ratios.

Since $r = 1$, the x-coordinate of $P(\theta)$ is equal to $\cos\theta$ and the y-coordinate to of $P(\theta)$ is equal to $\sin\theta$.

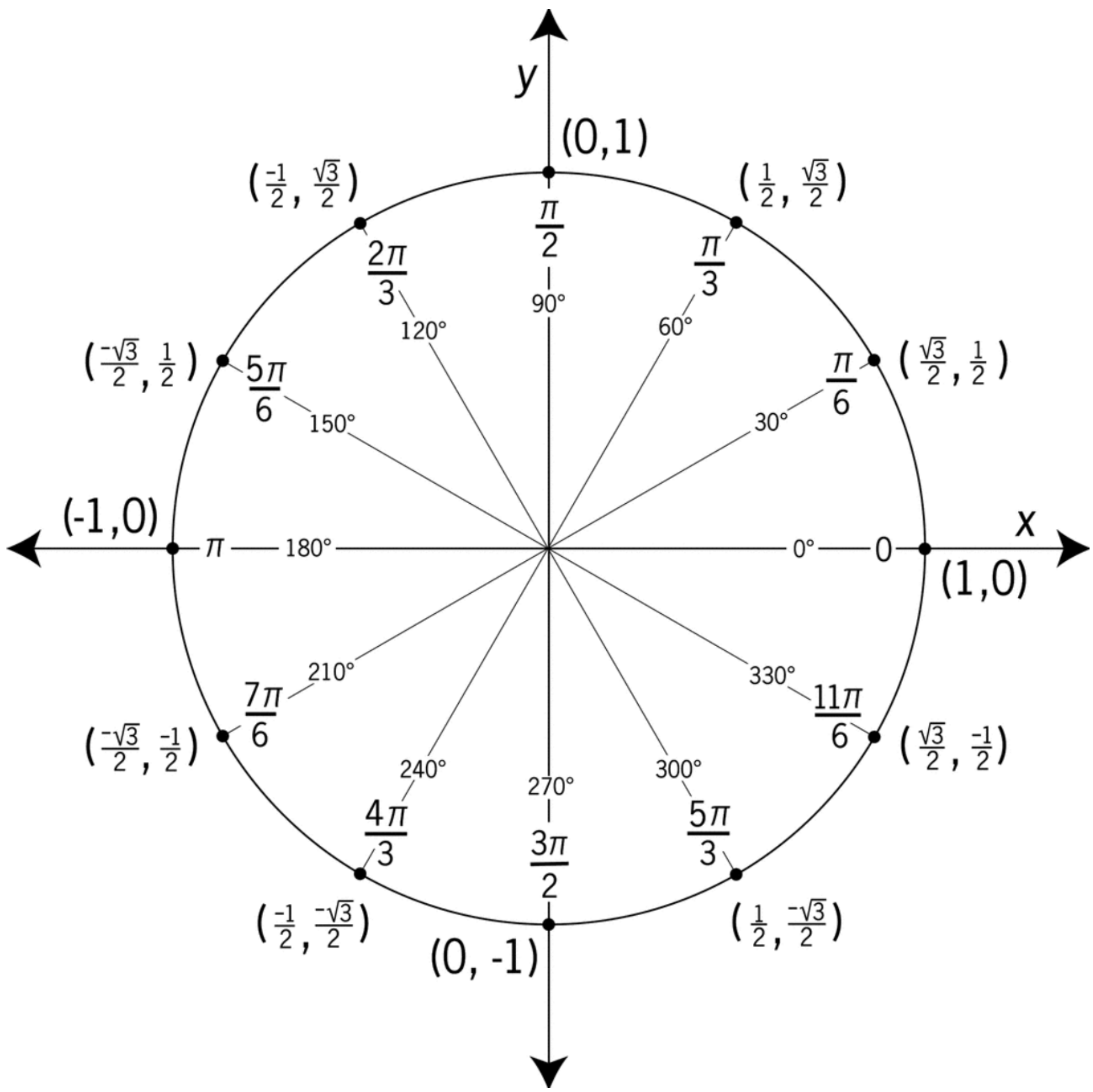
Why?



$$\cos 30 = \frac{\sqrt{3}}{2}$$

$$\sin 30 = \frac{1}{2}$$

THE UNIT CIRCLE



Ex: Find al possible values for the given trig points.

a) $P\left(\ ?,\frac{\sqrt{2}}{2}\right)$

$$P\left(?,\frac{\sqrt{2}}{2}\right)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\ and\ \left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

b) $P\left(\frac{7}{9},?\right)$

$$\cos\theta=\frac{7}{9}$$

$$o=\sqrt{9^2-7^2}$$
$$o=\sqrt{32}=4\sqrt{2}$$

$$\sin\theta=\frac{4\sqrt{2}}{9}$$

$$P\left(\frac{7}{9},?\right)=\left(\frac{7}{9},\frac{4\sqrt{2}}{9}\right)\ and\ \left(\frac{7}{9},\frac{-4\sqrt{2}}{9}\right)$$

Finding Values of θ

Ex: Given $0\ rad\le\theta$, determine the values of θ when $\sin\theta=\frac{\sqrt{2}}{2}$

Looking at the unit circle, we can see:

$$\theta_1=\frac{\pi}{4}\ and\ \theta_2=\frac{3\pi}{4}$$

But there is no upper limit given in the question, so we can go around the unit circle once and then get our angles, or twice and then get our angles, etc. So…

$$\theta_1=\frac{\pi}{4}+2\pi n\ and\ \theta_2=\frac{3\pi}{4}+2\pi n$$

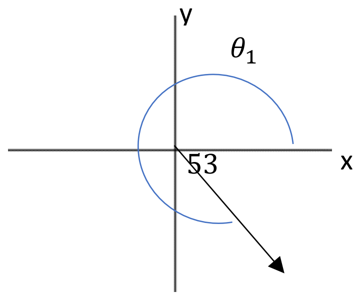
Where n is the number of additional rotations.

Ex: Given $P(\theta)=\left(\frac{3}{5},-\frac{4}{5}\right)$, solve for θ over $0\le\theta$

$$\sin\theta_R=\frac{4}{5}$$

Cos +, Sin -, so in Q4.

$$\theta_R=53$$



$$\theta_1=360-53$$

$$\theta_1=307$$

Since there is no upper limit on the restriction,

$$\theta_1=307+360n$$

Trig Identities

There are some trig expressions that are true, regardless of the value of the angle. We can use thee to simplify trig expressions, in proofs, and in solving complex trig equations.

There are 3 basic trig identities (and the variations of each).

Basic Identity	Variations
$sin^2\theta + cos^2\theta = 1$	$1 - sin^2\theta = cos^2\theta$
	$1 - cos^2\theta = sin^2\theta$
$1 + tan^2\theta = sec^2\theta$	$sec^2\theta - tan^2\theta = 1$
	$tan^2\theta = sec^2\theta - 1$
$1 + cot^2\theta = csc^2\theta$	$csc^2\theta - cot^2\theta = 1$
	$cot^2\theta = csc^2\theta - 1$

Proof: $sin^2\theta + cos^2\theta = 1$

Recall $sin = \frac{o}{h}\ and\ cos = \frac{a}{h}$

$$sin^2\theta + cos^2\theta = 1$$

$$\left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 = 1$$

$$\frac{o^2 + a^2}{h^2} = 1$$

Recall Pythagorean theorem:

$a^2 + b^2 = c^2$

$o^2 + a^2 = h^2$

$$\frac{h^2}{h^2} = 1$$

$$1 = 1$$

QED

QED is a Latin abbreviation for quod erat demonstrandum: “which was to be demonstrated”. It signifies that the argument has been proven.

Proof: $1 + tan^2\theta = sec^2\theta$

Recall $tan = \frac{o}{a}$ $and\ sec = \frac{h}{a}$

$$1 + tan^2\theta = sec^2\theta$$

$$1 + \left(\frac{o}{a}\right)^2 = \left(\frac{h}{a}\right)^2$$

$$1 = \left(\frac{h}{a}\right)^2 - \left(\frac{o}{a}\right)^2$$

$$1 = \frac{h^2 - o^2}{a^2}$$

Recall Pythagorean theorem:

$b^2 = c^2 - a^2$

$a^2 = h^2 - o^2$

$$1 = \frac{a^2}{a^2}$$

$$1 = 1$$

QED

Proof: $1 + cot^2\theta = csc^2\theta$

Recall $cot = \frac{a}{o}$ $and\ csc = \frac{h}{o}$

$$1 + cot^2\theta = csc^2\theta$$

$$1 + \frac{a^2}{o^2} = \frac{h^2}{o^2}$$

$$1 = \frac{h^2}{o^2} - \frac{a^2}{o^2}$$

$$1 = \frac{h^2 - a^2}{o^2}$$

Recall Pythagorean theorem:

$a^2 = c^2 - b^2$

$o^2 = h^2 - a^2$

$$1 = \frac{o^2}{o^2}$$

$$1 = 1$$

QED

We can use the trig identities above, as well as our trig ratios, to simplify trigonometric expressions.

Remember: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$ $\quad \sec\theta = \frac{1}{\cos\theta}$ $\quad \csc\theta = \frac{1}{\sin\theta}$

Ex: Simplify the following expressions.

a) $1 - cos^2\theta$

$$= sin^2\theta$$

b) $csc^2\theta - cot^2\theta$

$$= 1 + cot^2\theta - cot^2\theta$$

$$= 1$$

c) $\tan\theta\cos\theta$

$$= \frac{\sin\theta}{\cos\theta} \cdot \cos\theta$$

$$= \sin\theta$$

d) $(1 - cos^2\theta)(cot^2\theta)$

$$= (sin^2\theta)\left(\frac{cos^2\theta}{sin^2\theta}\right)$$

$$= cos^2\theta$$

e) $tan^2\theta\ csc\theta\ cos\theta$

$$= \left(\frac{sin^2\theta}{cos^2\theta}\right)\left(\frac{1}{sin\theta}\right)\left(\frac{cos\theta}{1}\right)$$

$$= \left(\frac{sin\theta}{cos\theta}\right)$$

$$= tan\theta$$

f) $csc\theta\sqrt{sec^2\theta - 1}$

$$= \left(\frac{1}{sin\theta}\right)\sqrt{tan^2\theta}$$

$$= \left(\frac{1}{sin\theta}\right) tan\theta$$

$$= \left(\frac{1}{sin\theta}\right)\left(\frac{sin\theta}{cos\theta}\right)$$

$$= \frac{1}{cos\theta}$$

$$= sec\theta$$

We can also use our trig identities to prove trig equations. To do this, we simplify the more complex side (generally the left side) to match the simpler side (generally the right side).

When proving trig equations, some things to consider:

- Factoring (especially using the difference of squares)
- Simplifying/combining fractions
- Align the = signs
- Typically work left to right
- Manipulate only one side

Ex: Prove the following

a) $tan^2\theta cos^2\theta + cos^2\theta = 1$

$$tan^2\theta cos^2\theta + cos^2\theta = 1$$

$$\left(\frac{sin^2\theta}{cos^2\theta}\right)cos^2\theta + cos^2\theta = 1$$

$$sin^2\theta + cos^2\theta = 1$$

$$1 = 1$$

QED

b) $sec\theta - cos\theta = sin\theta\ tan\theta$

$$sec\theta - cos\theta = sin\theta\ tan\theta$$

$$\frac{1}{cos\theta} - cos\theta = sin\theta\ tan\theta$$

$$\frac{1}{cos\theta} - \left(\frac{cos^2\theta}{cos\theta}\right) = sin\theta\ tan\theta$$

$$\left(\frac{1 - cos^2\theta}{cos\theta}\right) = sin\theta\ tan\theta$$

$$\left(\frac{sin^2\theta}{cos\theta}\right) = sin\theta\ tan\theta$$

$$sin\theta\left(\frac{sin\theta}{cos\theta}\right) = sin\theta\ tan\theta$$

$$sin\theta\ tan\theta = sin\theta\ tan\theta$$

QED

c) $sin^2\theta cot^2\theta sec\theta = cos\theta$

$$sin^2\theta cot^2\theta sec\theta = cos\theta$$

$$sin^2\theta\left(\frac{cos^2\theta}{sin^2\theta}\right)\left(\frac{1}{cos\theta}\right) = cos\theta$$

$$\left(\frac{cos^2\theta}{1}\right)\left(\frac{1}{cos\theta}\right) = cos\theta$$

$$cos\theta = cos\theta$$

QED