Graph Theory Unit – Definitions

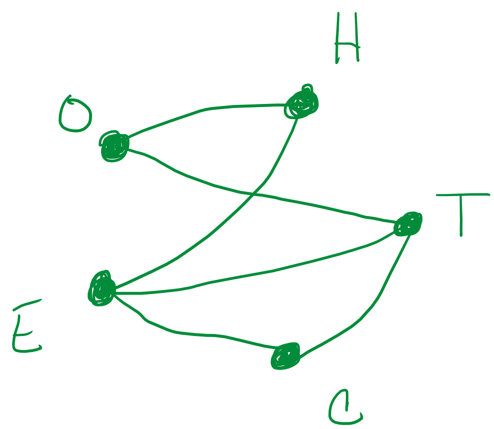
2.1 GRAPH THEORY DEFINITIONS

In graph theory, a graph is a collection of dots and lines. The lines show relationships that exist between the elements of a set (the dots).

Ex: An airline has flights to 5 different airports (Ottawa, Halifax, Toronto, Calgary, and Edmonton). The following is information about flights offered:

- From Ottawa, there are flights to Halifax and Toronto
- From Halifax, there are flights to Ottawa and Edmonton
- From Toronto, there are flights to Ottawa, Calgary, and Edmonton
- From Calgary, there are flights to Toronto and Edmonton
- From Edmonton, there are flights to Toronto, Halifax and Calgary

Draw a graph to represent this scenario.



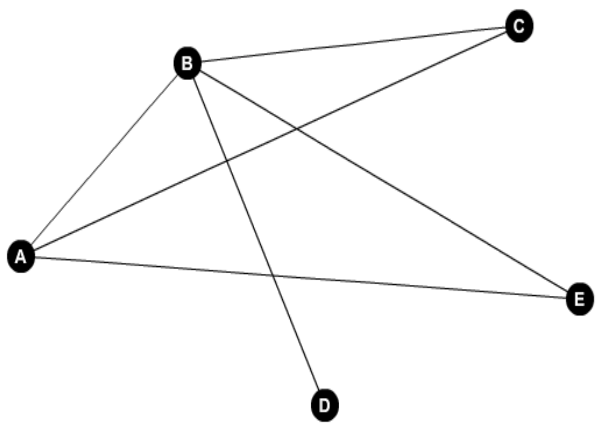
Practice Question

1) A company is installing a new computer network and some computers have to be directly connected to others.

- There are 6 computers (A, B, C, D, E, F)
- A must be directly connected to B, C, E, F
- B must be directly connected to A, C and D
- C must be directly connected to A, B
- D must be directly connected to B and E
- E must be directly connected to A and D
- F must be directly connected to A

Draw a graph to represent this scenario.

Graph Theory Unit – Definitions



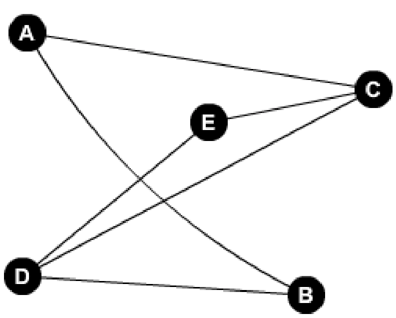
- Vertex – a dot on the graph. The vertices are the elements of the set (people, places, etc.). They are typically labelled with letters.
  A, B, C, D, E
- Edge – a line on the graph. The edges represent relationships between the vertices. Edges are labelled using the letters of the two vertices they connect. It does not matter which vertex is listed first. AB, AC, AE, BC, BD, BE
- Order – the number of vertices in a graph. 5
- Degree – the number of edges that touch the vertex.

A: 3 B: 4 C: 2

D: 1 E: 2

Practice Question

2) In the graph below, identify all the vertices and determine the degree of each vertex, identify all edges, and determine the order of the graph.

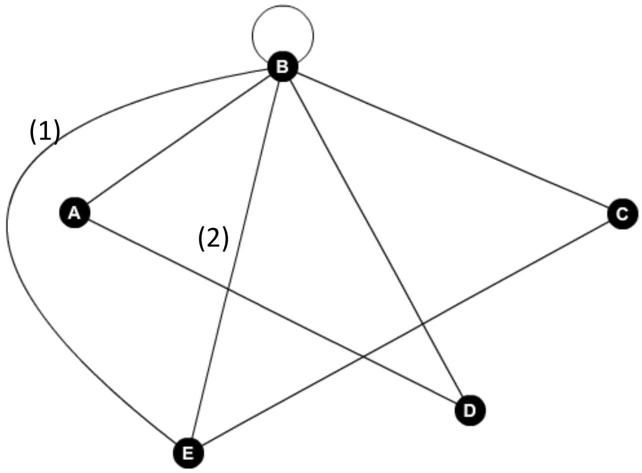


Vertices (and degree of each):

Edges:

Order:

Graph Theory Unit – Definitions



- Loop – an edge that starts and ends at the same vertex.

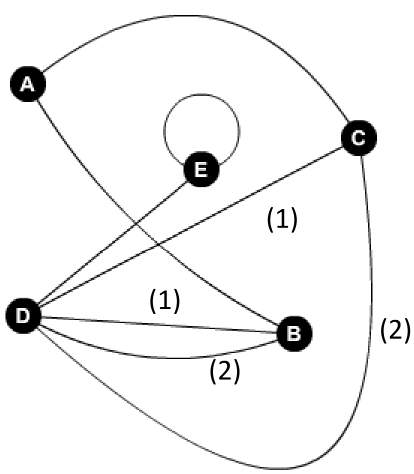
BB

- Parallel edges – when two or more edges connect the same two vertices.

BE (1)
BE (2)

Practice Question

3) In the graph below, identify any loops and parallel edges

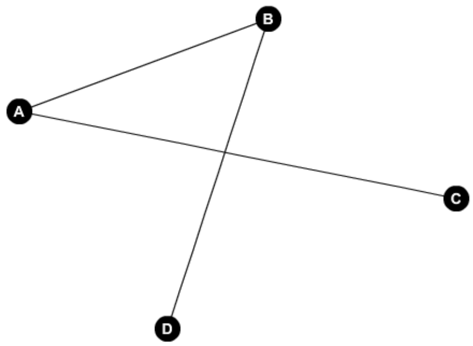


Loop(s):

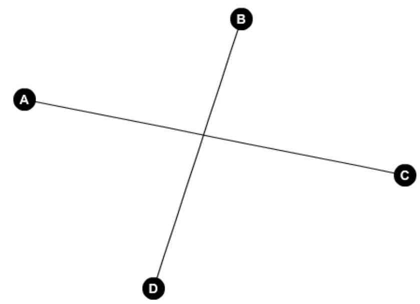
Parallel Edge(s):

Graph Theory Unit – Definitions

- A graph is **connected** when each vertex is connected to every other vertex by an edge *or by a series of edges*. This means that you can start at one vertex and get to every other vertex.

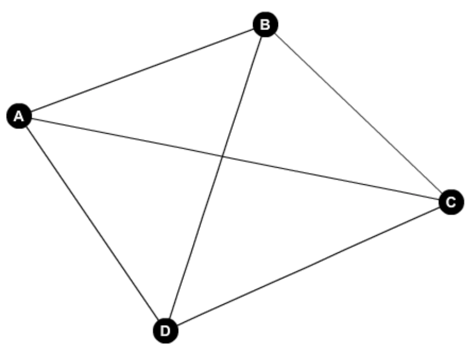


Connected

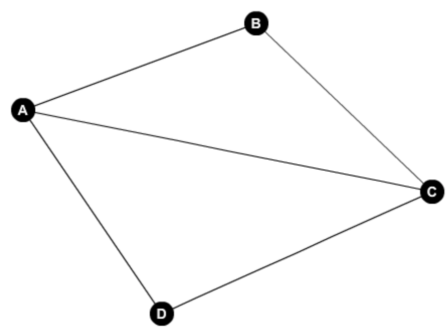


Not Connected

- A graph is **complete** when an edge connects every pair of vertices. This means there is only one edge separating every pair of vertices.



Complete

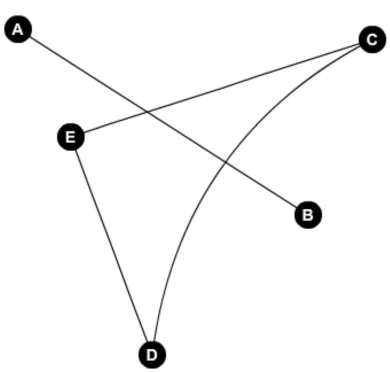


Not Complete

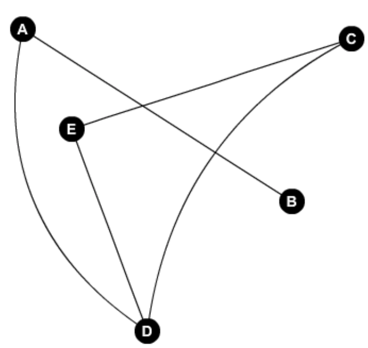
Practice Question

4) Determine if the graphs below are connected, complete, or neither.

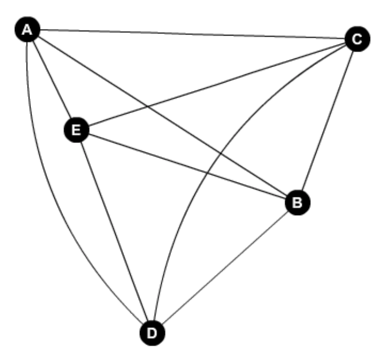
a)



b)

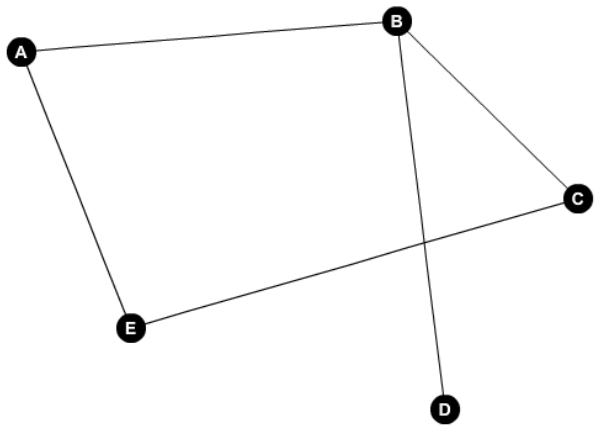


c)

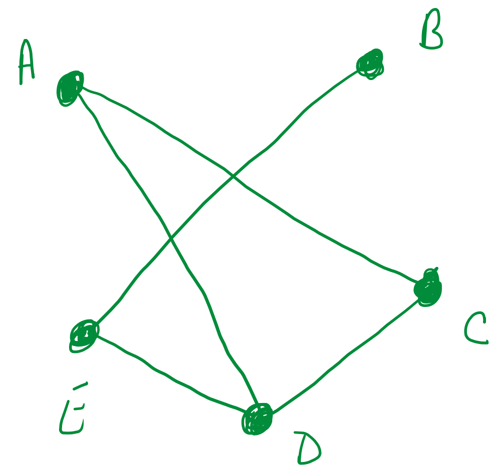


Graph Theory Unit – Definitions

- The **complement** of a graph is a graph that contains the edges necessary to complete the initial graph. You can think about this as the “missing edges”.



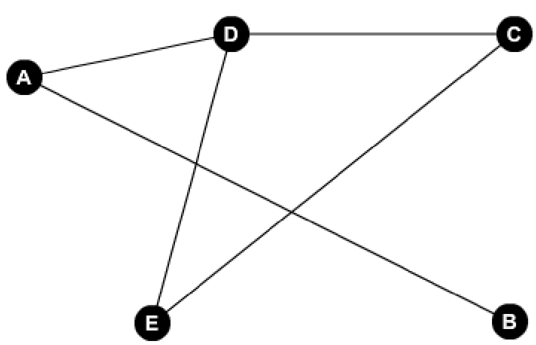
A graph



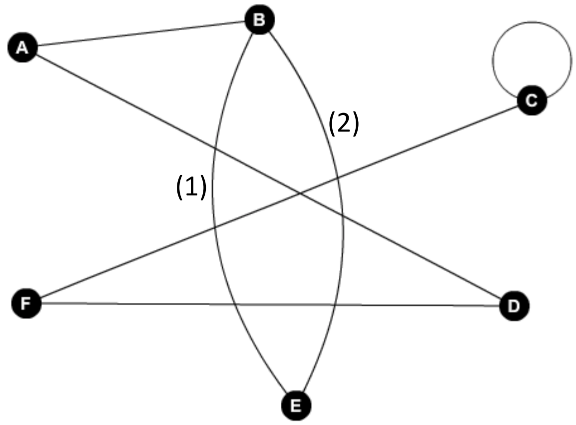
Draw the complement

Practice Questions

5) Create the complement of the graph given below.



6)



a) List all vertices and the degree of each

b) List all edges

c) Determine the order of the graph

d) Identify any loops

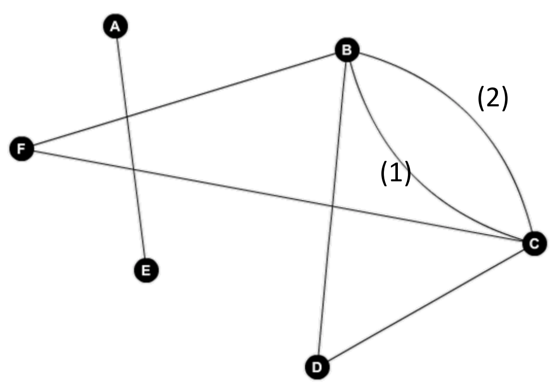
e) Identify any parallel edges

f) Determine whether the graph is connected

g) Determine whether the graph is complete

h) Draw the complement of the graph

7)



a) List all vertices and the degree of each

b) List all edges

c) Determine the order of the graph

d) Identify any loops

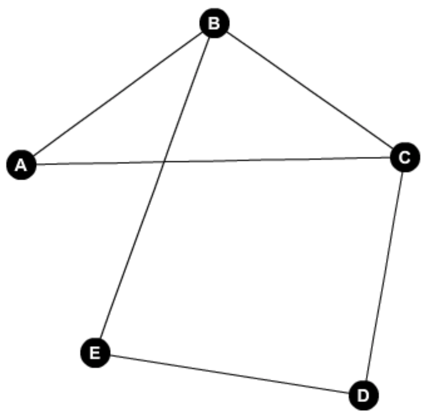
e) Identify any parallel edges

f) Determine whether the graph is connected

g) Determine whether the graph is complete

h) Draw the complement of the graph

2.2 GRAPH THEORY PATHS AND CIRCUITS



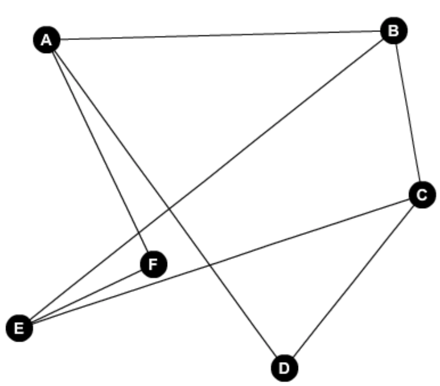
- A **path** on a graph is how we can get from one vertex to another. Vertices and edges can be repeated.
  On the graph above, an example of a path is ACD EDC
- A **simple path** is a path where no edges are repeated (vertices can be repeated).
  On the graph above, an example of a simple path is ACDE
- The **length** of a path is the number of edges it contains.
  On the graph above, the length of path ABCDEBAC is: 7

  On the graph above, the length of path CBEDCA is: 5
- The **distance** between two vertices is the length of the shortest path joining the vertices.
  On the graph above, $d(A, E)$ is: 2

  On the graph above, d(A,C) is: 1

Practice Question

1) On the graph below:



a) Name a path

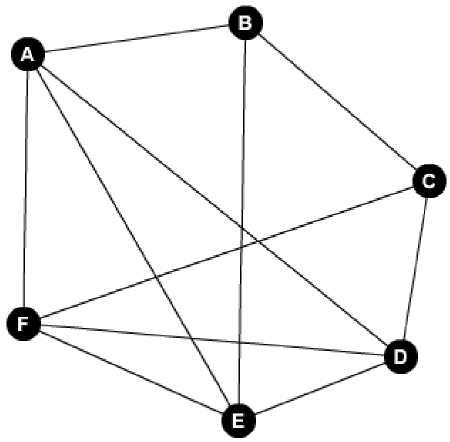
b) Name a simple path

c) What is the length of path DABEFAB?

d) Name a simple path with length 4

e) What is the distance between C and A?

f) What is the distance between A and D?



- A **circuit** is a path that starts and ends at the same vertex.

On the graph above, an example of a circuit is: C F D E F C

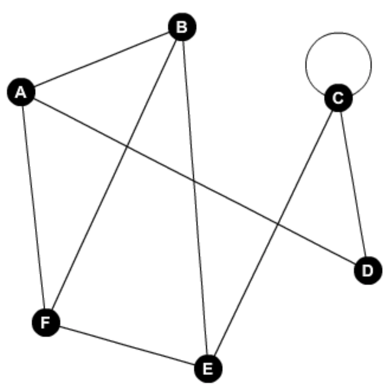
- A **simple circuit** is a circuit that has no repeated edges (but it does not need to use every edge)

On the graph above, an example of a simple circuit is:

C F A D C

Practice Questions

2) On the graph below:



a) find a path with length 4

b) find a simple path with length 6

c) find a circuit with length 5

d) find a simple circuit with length 3

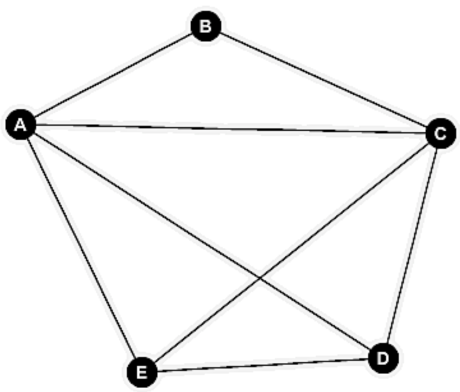
e) find the distance between vertices A and C

Graph Theory Unit – Euler Path and Circuit

2.3 EULER PATH AND CIRCUIT

- An **Euler path** (pronounced "oiler") is a path that travels over every edge once and only once.
  - An Euler path exists if exactly 2 vertices have a degree that is an odd number (and the rest have a degree that is an even number).
    - The Euler path will start at one of the vertices with an odd degree and end at the other vertex with an odd degree.
  - An Euler path also exists if all vertices have a degree that is an even number (but this is a special case, called an Euler circuit – see below)

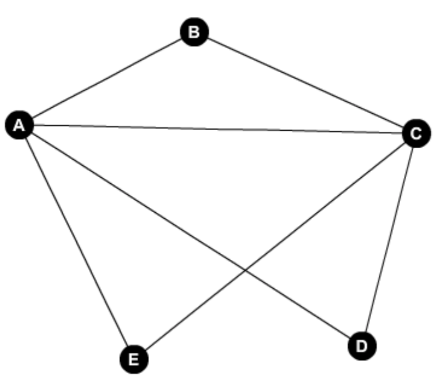
Ex: Find an Euler path in the graph below



DEABCADCE

- An **Euler circuit** is a circuit that travels over every edge once and only once (it is an Euler path that starts and ends at the same vertex).
  - An Euler circuit is a special case of an Euler path. It is an Euler path that begins and ends at the same vertex.
  - An Euler circuit exists if all vertices have degrees that are even numbers.

Ex: Find an Euler circuit in the example below

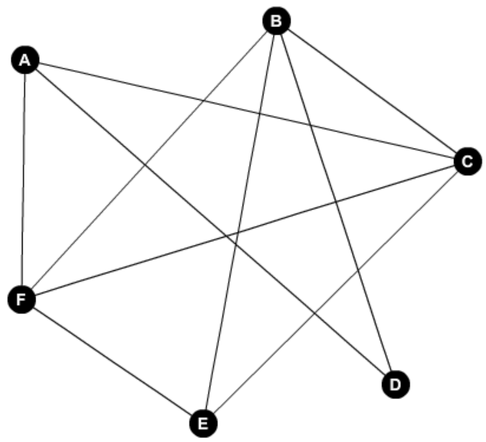


ABCADCEA

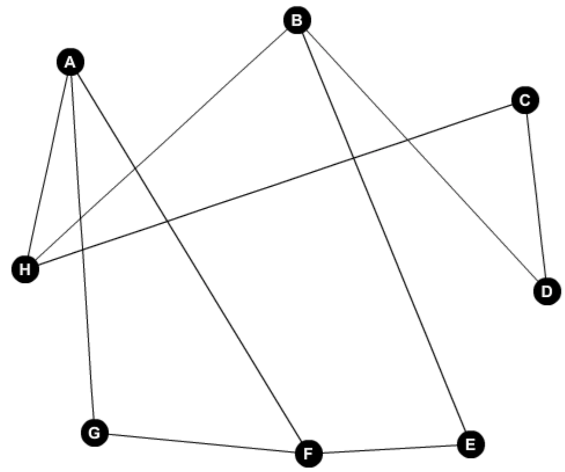
Practice Questions

1) Does an Euler path exist in the graphs below? If yes, identify one.

a)

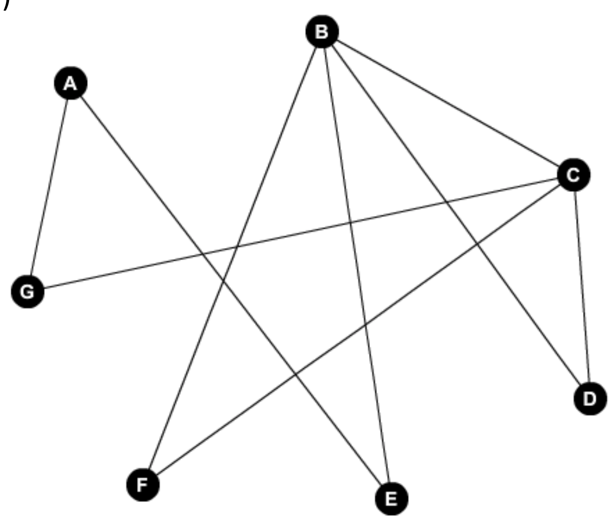


b)

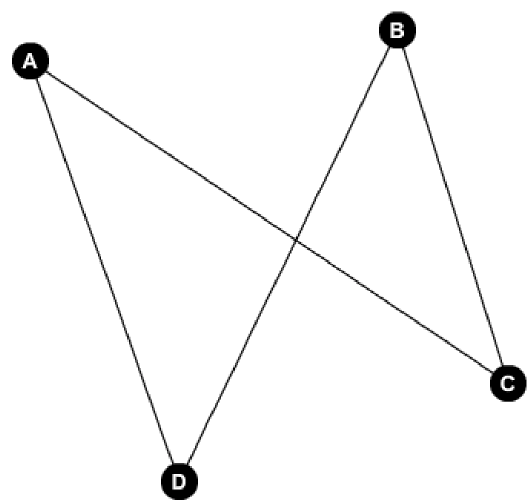


2) Does an Euler circuit exist in the graphs below? If yes, identify one.

a)



b)

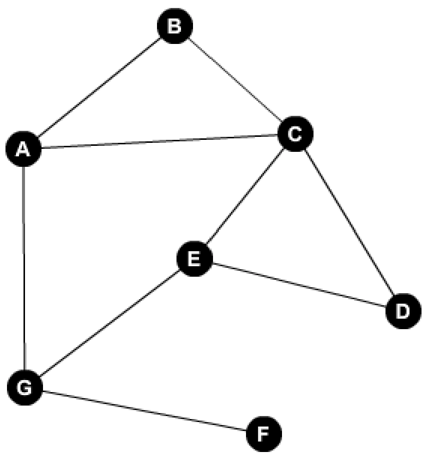


Graph Theory Unit – Hamiltonian Path and Circuit

2.4 HAMILTONIAN PATH AND CIRCUIT

- A **Hamiltonian path** is a path that passes through every vertex once and only once (it does not need to include every edge).
  - There is no good way to determine whether a Hamiltonian path exists other than trying to find one.

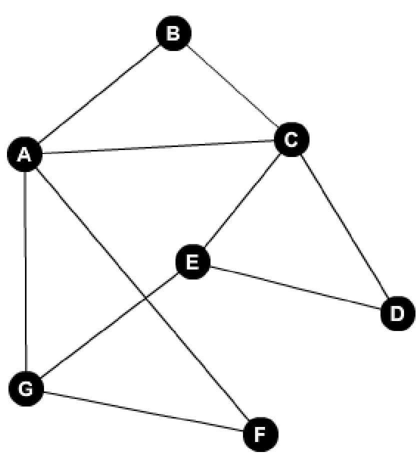
Ex: Find a Hamiltonian path in the graph below



FGABCDE

- A **Hamiltonian circuit** is a Hamiltonian path that begins and ends at the same vertex (so the first vertex is repeated as the last vertex, but no other vertex is included).

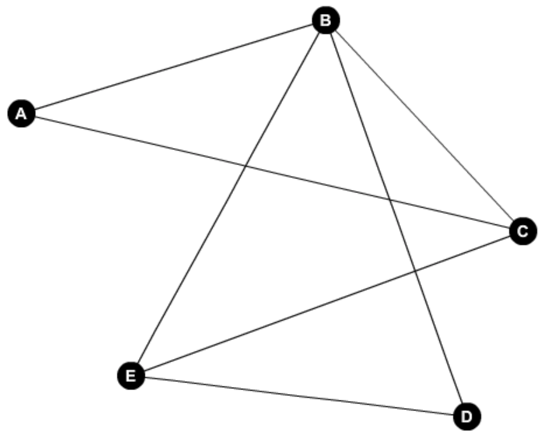
Ex: Find a Hamiltonian circuit in the graph below



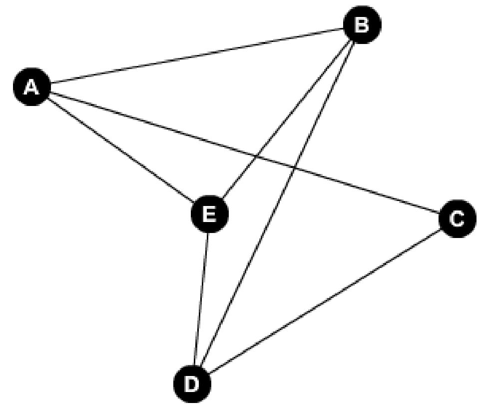
GFABCDEG

Practice Questions

1) Identify a Hamiltonian path in the graph below.

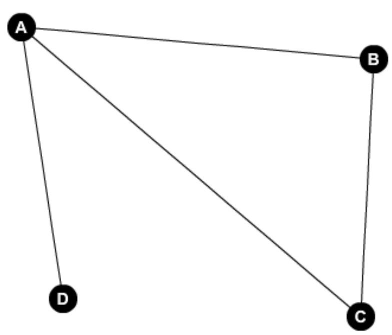


2) Identify a Hamiltonian circuit in the graph below.

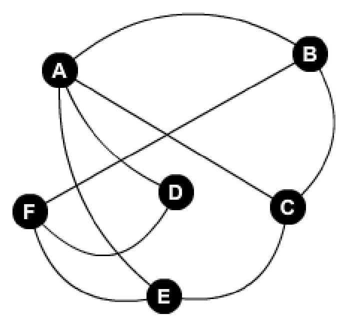


3) In each of the following graphs, indicate whether the following exist:
i) Euler path; ii) Euler circuit; iii) Hamiltonian path; iv) Hamiltonian circuit

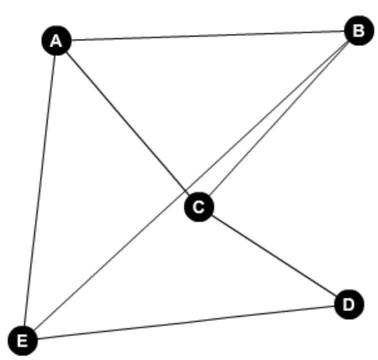
a)



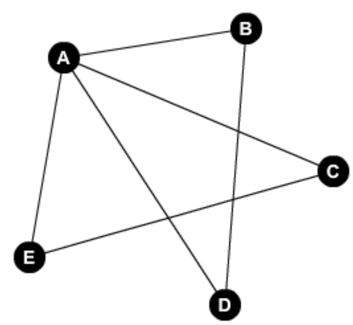
c)



b)



d)

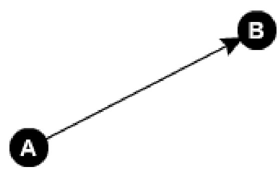


Graph Theory Unit – Directed Graphs

2.5 DIRECTED GRAPH

- A **directed edge** is like a one-way street – we can only move along that edge in one direction. It is represented with an arrow on the edge.
- A **directed graph** has one or more directed edges.

Ex:

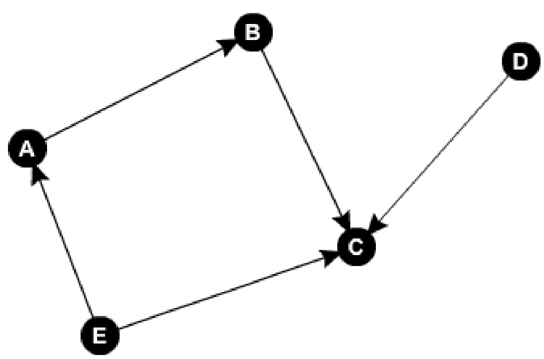


We can go from vertex A to vertex B, but not from vertex B to vertex A.

When we name directed edges, order matters.

This edge is AB (and not BA)

Ex:

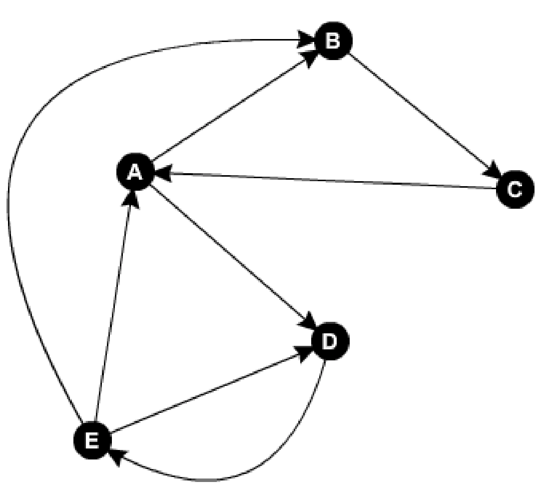


In this example, a path is ABC, but not ABCD.

This example does not have any circuits.

Practice Questions

1) In the graph below:



a) Identify a path of length 4.

b) Identify a simple circuit of length 3.

c) Does path A-E-D exist? Explain.

d) What is the length of path A-B-C-A-D-E?

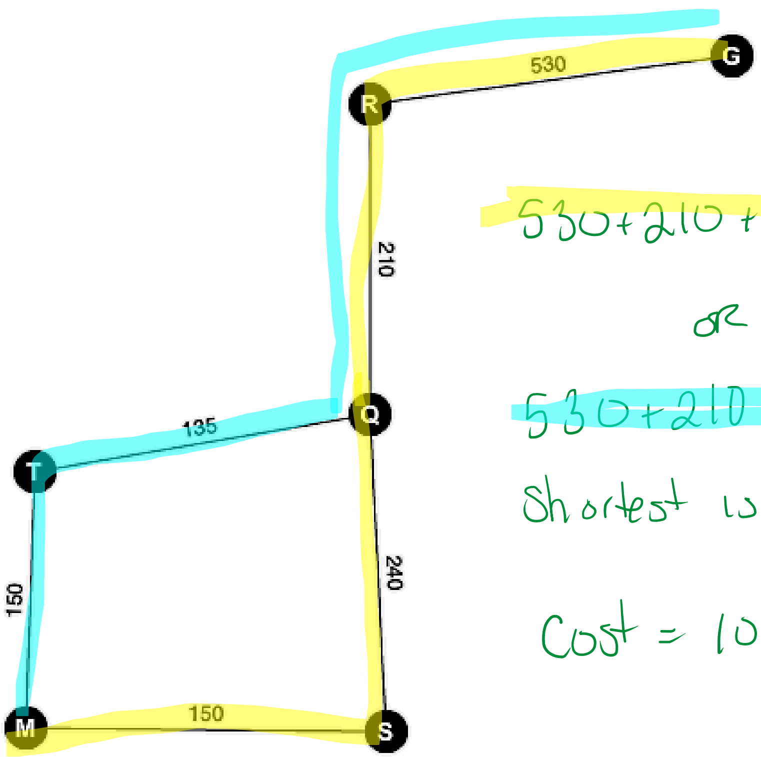
e) Determine $d(A, C)$.

Graph Theory Unit – Weighted Graphs

2.6 WEIGHTED GRAPHS (NETWORKS)

- A **weight** is a number, which could represent distance, time, etc.
- A **weighted graph** is a graph in which every edge is assigned a weight.
- The **weight of a path** is found by adding the weights of all the edges included in the path.
- A weighted graph can be directed or not.

Ex: The network below shows the distance (in km) between towns linked by a rail network.
Transporting heavy equipment by train costs approximately $100 per km.
Give the approximate cost of transporting heavy equipment from G to M if the train takes the shortest route.



530 + 210 + 240 + 150 = 1130 km

or

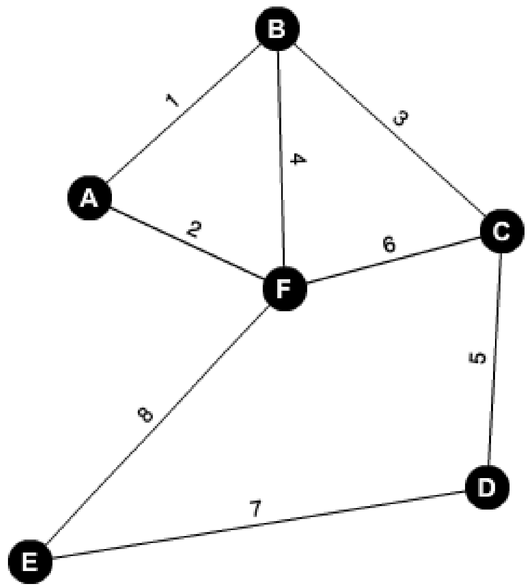
530 + 210 + 135 + 150 = 1025 km

Shortest is 1025 km

Cost = 1025 km x $100 = $102,500

Practice Question

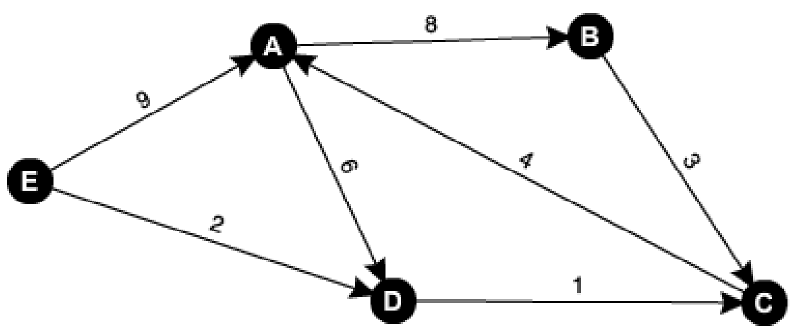
1) Based on the graph below, determine:



a) The weight of path A-B-C-F

b) The weight of circuit C-D-E-F-C

2) The graph below is weighted and directed.



a) Determine the weight of path E-D-C-A-D

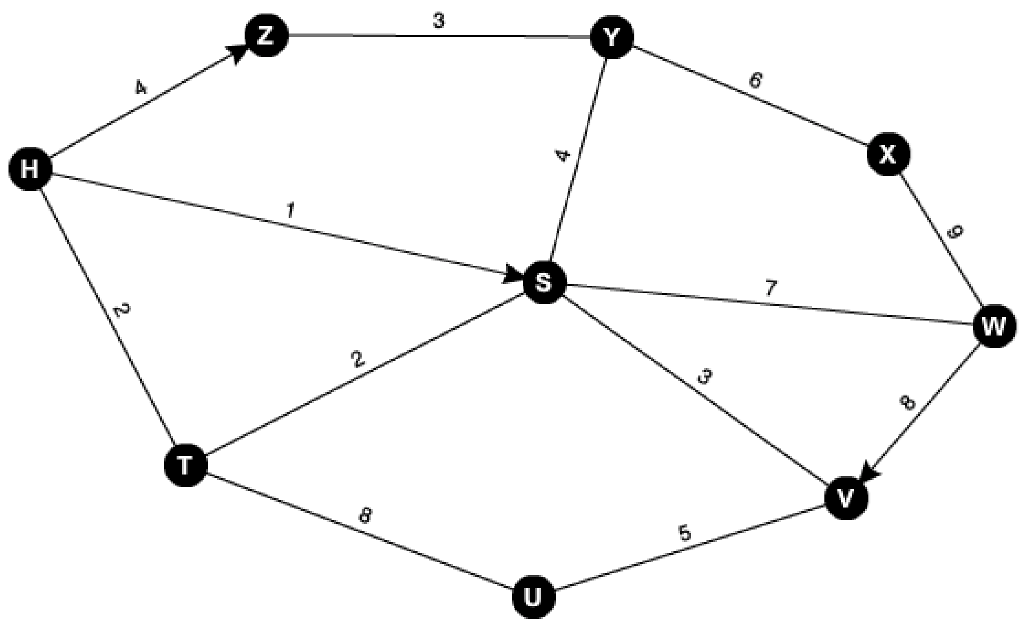
b) Determine the weight of path B-C-A-D

c) Determine the weight of path E-A-B-C

d) Identify all the simple paths going from vertex E to vertex B

e) Of the paths identified in part d, which one has the smallest weight?

3) The following graph presents the length (in km) of various routes Louis could take as he trains for a marathon. He begins and ends each training session at his home, H.



a) For easy runs, Louis wants to run less than 10km without taking the same road twice. What route can he take and how long is his run?

b) How long is Louis's run if he takes the following route: H-Z-Y-S-V-U-T-H?

c) Louis would like to run a circuit that covers between 30 and 35 km without taking the same road twice. Is this possible (and if yes, what route could he take)?

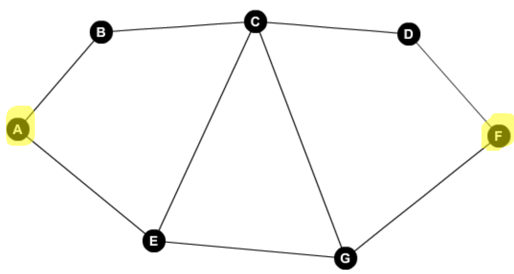
Graph Theory Unit – Path of Optimal Value

2.7 PATH OF OPTIMAL VALUE

- When we want to find the **path of optimal value**, we are looking for the longest or shortest path (the maximum or minimum, depending on the question) from one vertex to another.
  - On a graph without weighted edges, the **path of minimum value** between two vertices corresponds to the path using the fewest edges between those vertices.
  - On a graph without weighted edges, the **path of maximum value** between two vertices corresponds to the simple path (no repeated edges) using the most edges between those vertices.
  - On a weighted graph, the **path of minimum value** between two vertices corresponds to the path that has the smallest weight between those vertices.
  - On a weighted graph, the **path of maximum value** between two vertices corresponds to the simple path (no repeated edges) with the largest weight between those vertices.

In the graphs below, determine the path of minimum value and the path of maximum value with an initial vertex A and a final vertex F.

a)



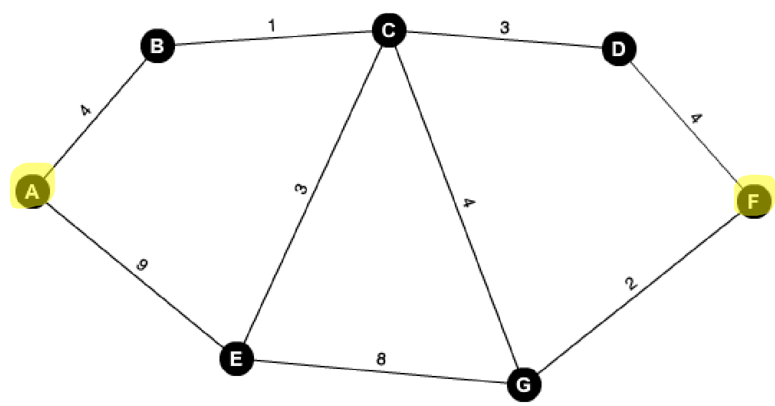
Path of minimum value:

AEGF = 3

Path of maximum value:

ABCEGCDE = 7

b)



Path of minimum value:

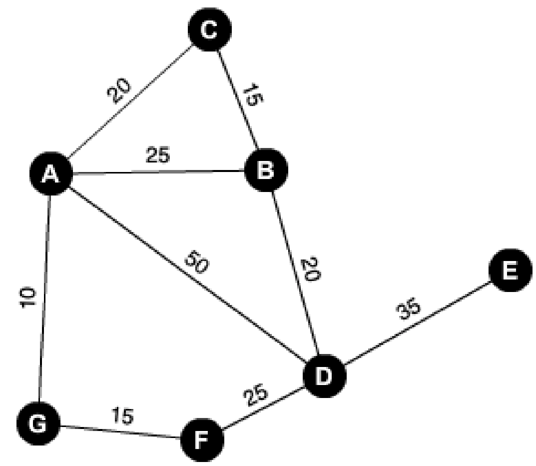
ABCGF = 11

Path of maximum value:

AEGCDF = 28

Practice Question

1) In the graph below, the values represent the amount (in thousands of dollars) that a construction company must pay to transport its equipment from one city to another.



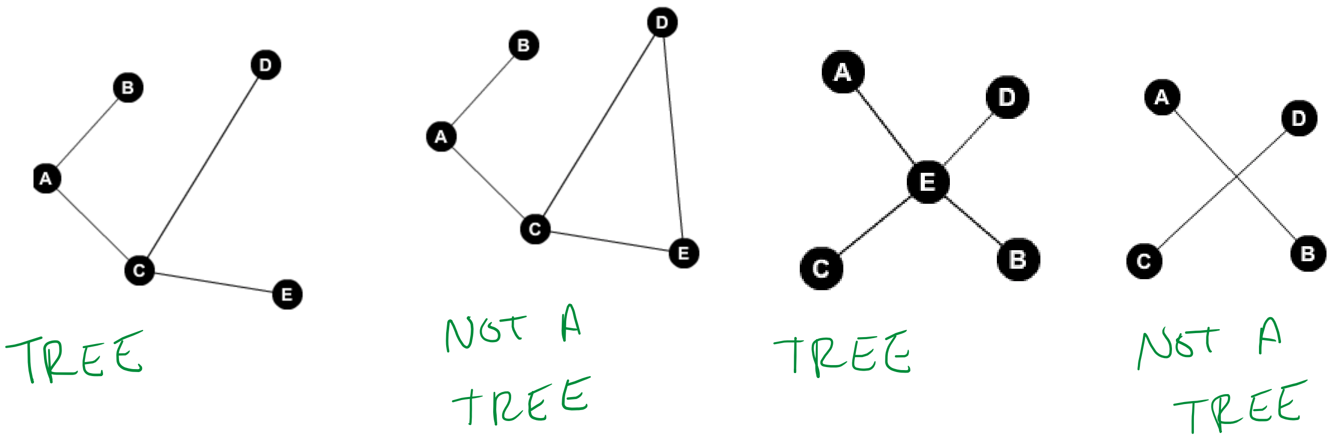
a) Determine the minimum transport cost from City A to City E

b) Determine the minimum transport cost from City G to City B

2.8 TREE OF OPTIMAL VALUE

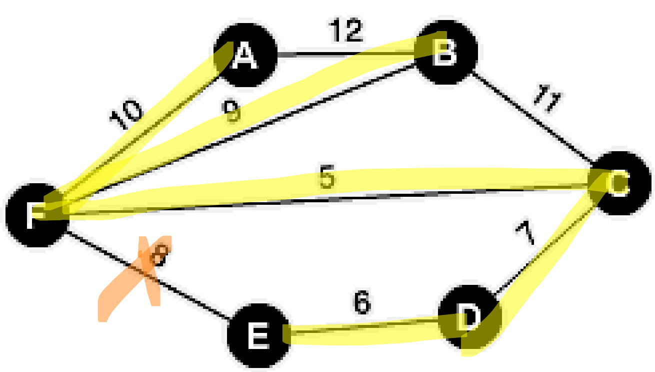
- A **tree** is a connected graph with no simple circuits

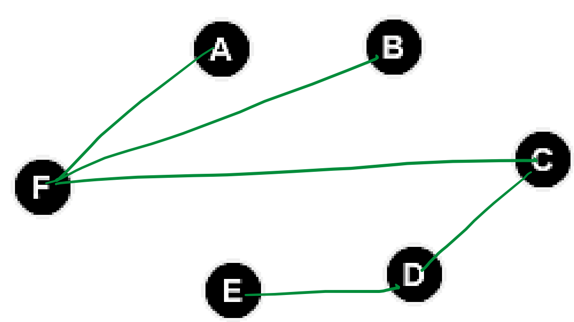
Examples:



- A **tree of optimal value** selects only some edges in a graph such that all vertices are connected, there are no simple circuits, and we use either the largest or smallest weighted edges.
- There are several steps to creating a tree of optimal value. If we are looking for a tree of minimum value:
    - Copy the vertices of the graph
    - Select the edge with the lowest value and draw it.
    - Of the remaining edges, select the one with the lowest value and draw it.
    - Keep selecting the smallest edge and drawing it, unless an edge will create a simple circuit. Then skip that edge and move to the next one. Once you have created a tree, stop drawing edges.
- To create a tree of maximum value, follow the steps above, but use edges with the largest value.

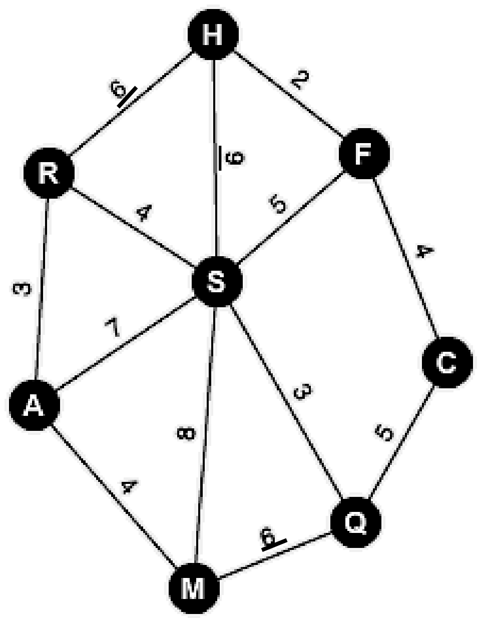
Ex: Create a tree of minimum value given the graph below





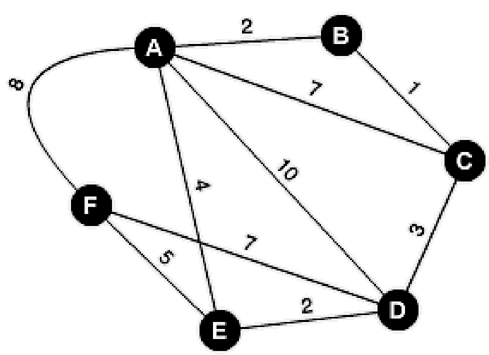
Practice Questions

1) A hydroelectric dam generates electricity which is transmitted using high-tension lines to Quebec's main urban centers. These transmission lines lose some of their power as the electricity is being transmitted. The graph below shows some of these lines and the power loss in tens of megawatts. Vertex H is the hydroelectric dam.



The facility's engineering crew wants to minimize the power loss while connecting all the urban centers.
Which high-tension lines should be used and what is the total power loss?

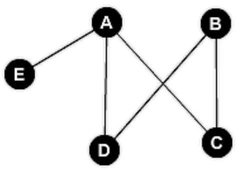
2) A city park has developed a series of trails between various parking areas. In the winter, the city wants to maintain the fewest number of trails such that all parking areas are still connected, either directly or indirectly, but the total trail distance is maximized. The graph below represents the parking areas (vertices) and trails (edges) where the weight of each edge represents the length (in km) of that trail.



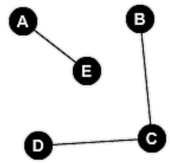
Which paths should the city maintain in the winter and what is the total distance of trails available?

3) In the following graphs, what edge(s) would need to be added or removed in order for the graph to be a tree?

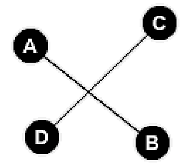
a)



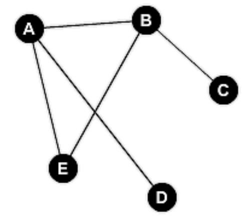
b)



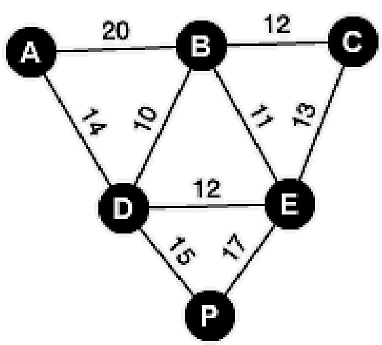
c)



d)

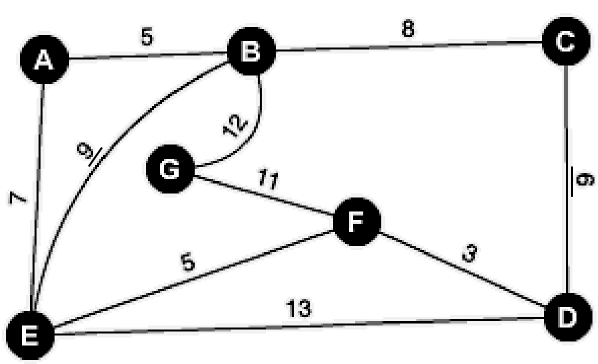


4) The sewage lines in a city need an upgrade and every neighborhood must be connected (either directly or indirectly) by sewage line to the purification plant, P. The graph below represents the neighborhoods in a city and the cost of upgrading the sewage lines (in thousands of $).



What is the minimum cost for the city to upgrade the sewage lines?

5) The cost of hiring a taxi is based on the amount of time it takes to get from your starting place to your destination. The graph below represents a taxi ride's travel time (in min) between intersections.



If a taxi costs $1.50 per minute, calculate:

a) The minimum cost to get from Point F to Point B

b) The minimum cost to get from Point A to Point D

Graph Theory Unit – Critical Path

2.9 CRITICAL PATH

If we have many tasks that are necessary to complete a project, and some of the tasks can be carried out simultaneously, we can use graph theory to determine the minimum amount of time necessary to complete the project.

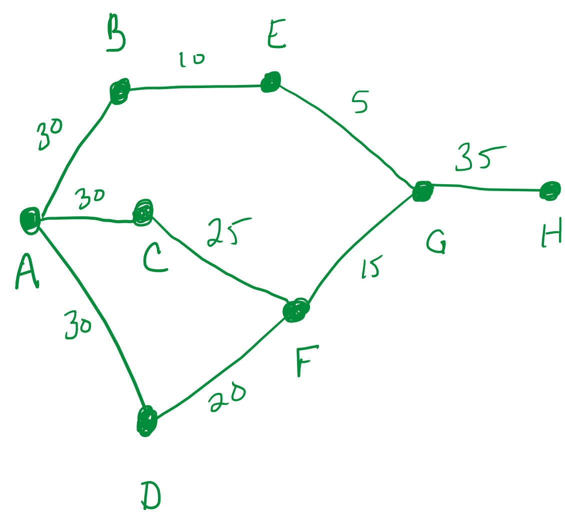
- A **critical path** (the minimum amount of time to complete all steps) is the simple path of ***maximum value***.

There are 5 keys to determining a critical path

1. Each vertex is a step in the project.
2. Each edge is weighted with the time it takes to complete the step (using the vertex at the start of the line) and each edge is directed.
3. Use the steps and time to create a graph.
4. Find all possible simple paths from the start of the project to the end.
5. Choose the *longest* path.

Ex: There are several steps required to launch a business. The following table lists the necessary steps, the time it takes to complete each step, and provides information on whether there are any steps that need to be completed before a step can begin.

Step	Description	Execution Time (in days)	Prior Steps
A	Prepare a business plan	30	None
B	Conduct market research	10	A
C	Find partners	25	A
D	Find location	20	A
E	Analyze market research	5	B
F	Evaluate product-distribution system	15	C and D
G	Arrange financing	35	E and F
H	Launch company	None	G



ABEGH = 80
ACFGH = 105 *
ADFGH = 100

Determine the minimum time necessary to complete all the steps.

The minimum time necessary is 105 days.

Practice Questions

1) An accounting firm hires a company to review its computer system. The following is the company's proposal for the implementation of a new system.

Step	Description	Execution Time (days)	Prior Steps
A	Analysis of needs	10	None
B	Detailed analysis of project	8	A
C	Purchase of computers	21	B
D	Training of programming team	3	B
E	Designing the accounting system	6	C
F	Coding the accounting system	18	C
G	Updating the computers	5	E and F
H	Installation and delivery of the computers and accounting system	4	D and G
I	End of computer work	None	H

Determine the minimum amount of time necessary for the implementation of this new system.

2) Tyson is having friends over for dinner. He starts preparing the meal at 3:45 pm. The steps required for the dinner preparation are as follows:

Step	Description	Execution time (min.)	Prior Steps
A	Choose the menu	5	None
B	Peel the carrots	5	A
C	Prepare the meatballs	15	A
D	Prepare the sauce	5	B
E	Cook the meatballs in the sauce with the carrots	45	C and D
F	Prepare the salad	10	D
G	Prepare the appetizer	15	C
H	Cook the appetizer	15	G
I	Prepare the dessert	20	G
J	Set the table	10	E and F
K	Serve dinner	5	H, I, J
L	End of preparation	None	K

Tyson would like to complete all the steps by 5:00pm. Is this possible? Explain.

3) The process of writing, giving, and analysing tests teachers give to a class requires several steps, some of which can be carried out simultaneously. The steps required are as follows:

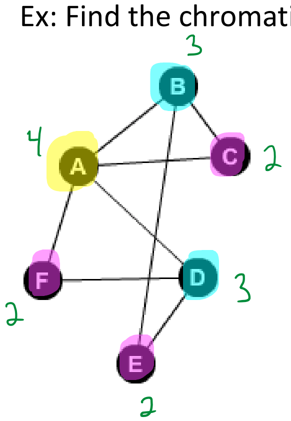
Step	Description	Time (days)	Prior Steps
A	Review the material	1	None
B	Decide on structure of exam	1	None
C	Write the questions	5	A and B
D	Review the questions	2	C
E	Write the exam	1	D
F	Check the exam	1	E
G	Give the exam	1	F
H	Correct the exam	2	G
I	Analyse the results	1	G
J	End of task	None	H and I

What is the minimum amount of time necessary to complete this project?

2.10 CHROMATIC NUMBER

- The **chromatic number** of a graph is the fewest number of colors required to color all the vertices of a graph while making sure no two adjacent vertices (vertices directly connected to one another) are the same color.
- Chromatic number can be used to:
  - Categorize information into as few groups as possible (ex: the number of colors required to color a map so that no countries sharing a border are the same color).
  - Categorize or group data based on incompatibilities (ex: making groups so that people who do not work well together are in different groups).
- There are 4 steps to determining the chromatic number of a graph
  1. Find the vertex with the highest degree and color it (if there is a tie, start with either)
  2. Find the vertices adjacent to the one you colored. Starting with the highest degree of those, color them a different color than the first vertex. Use the same color for all, unless they are adjacent. Use as few colors as possible.
  3. Of the remaining vertices, start with the highest degree and color is so that it is not the same color as any vertex it is adjacent to, using colors already used, if possible.
  4. The number of colors used is the chromatic number

Ex: Find the chromatic number of the graph below

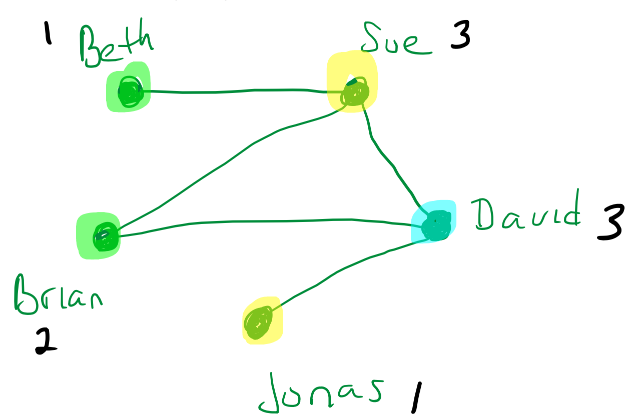


Chromatic number is 3

Ex: You are the team leader and have a group of 5 people. You want to make small groups to complete a task. It does not matter if the groups are the same size, but you must make sure people who do not work well together are in different groups.

- Beth cannot work with Sue
- David and Jonas argue so will not get any work done
- Sue and Brian are unproductive together
- David and Sue are poorly matched
- Brian and David are not compatible

What is the fewest number of groups you can make?

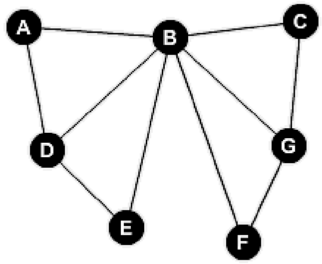


The fewest number of groups is 3

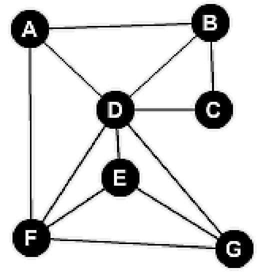
Practice Questions

1) Determine the chromatic number for the graphs below.

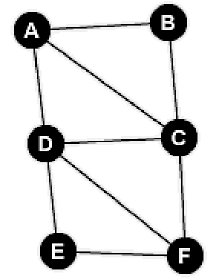
a)



b)



c)



2) Lydia has to color the map of South America for her geography class. The teacher has asked students not to use the same color for two countries that share a common border. Lydia only has 4 colors she can use and she wants to know if she can complete this task.

Using the map:

a) Construct a graph (vertices are the countries and edges represent the countries that share a common border).

b) Determine the chromatic number

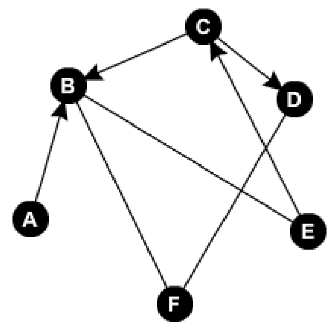
c) Can Lydia complete the task?

Graph Theory Unit – Exam Style Questions

2.11 EXAM STYLE QUESTIONS

Multiple Choice

1) Consider the graph below.



Which of the following does this graph contain?

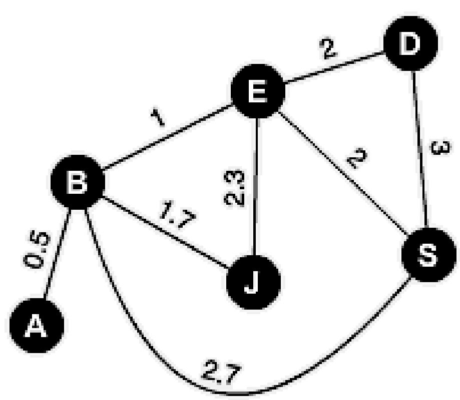
A) Euler Path C) Hamiltonian Path

B) Euler Circuit D) Hamiltonian Circuit

2) Emily (E), Jake (J), Sevanna (S), and Durham (D) are sharing a taxi to the airport (A). Vertex B is an intersection.

The taxi will start at Emily's house and then will pick up Jake and Sevanna in any order.
They will then drive to pick up Durham before heading to the airport.

The graph below represents a map of the possible routes the taxi can take with the weights representing the distance (in km). The taxi can use the same edge more than one.



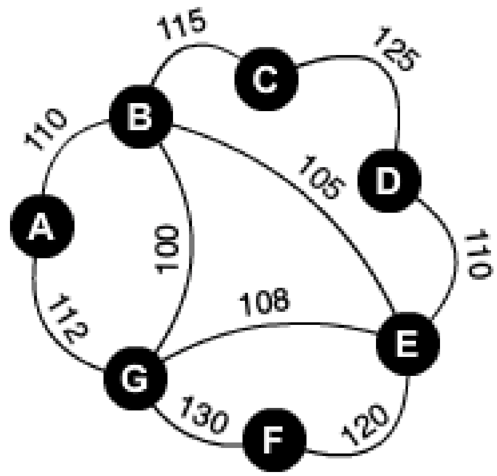
What is the length of the shortest route the taxi can take?

A) 10.5 km C) 13.1 km

B) 12.6 km D) 14.1 km

3) Roger wants to install a watering system in his garden. On the graph below:

- The sprinklers are represented by vertices A, B, C, D, E, F, and G.
- The pipes connecting the sprinklers are represented by the edges.
- The number associated with each edge represents the cost of installation, in dollars.



What is the lowest cost to connect all the sprinklers while minimizing the cost?

A) $313

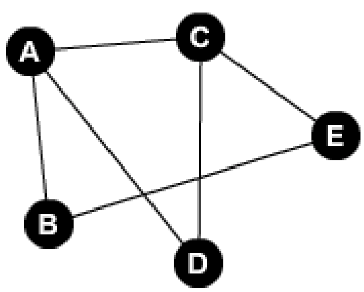
B) $660

C) $822

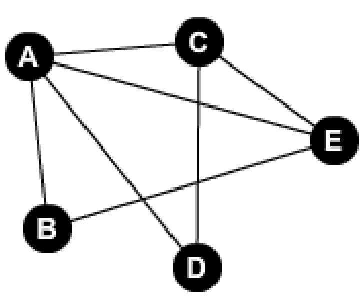
D) $880

4) **Which of the following graphs contains and Euler circuit?**

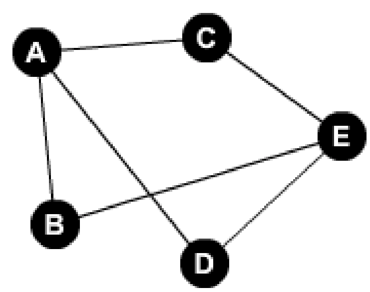
A)



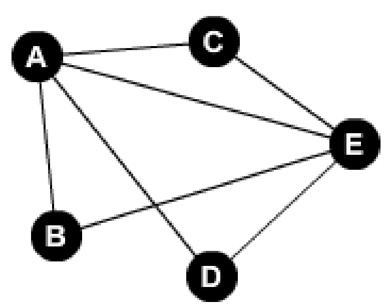
B)



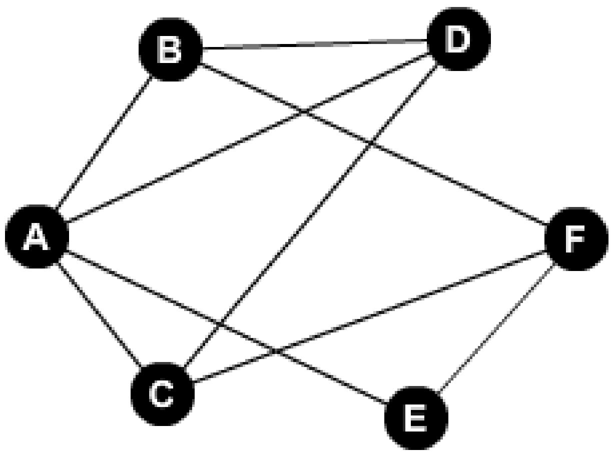
C)



D)



5) **What is the chromatic number for the graph below?**



A) 2

B) 3

C) 4

D) 5

Short Answer

6) Dr. James is dividing her class into groups and she wants to make sure all students in a group work well together. Dr. James wants to create the fewest number of groups and wants to make sure every student is in a group with at least one other student.

The table below contains information about students who do not work well together.

Student	Is incompatible with…
Amanda	Chris, David, Francis
Bob	David, Francis
Chris	Amanda, Eli, Francis
David	Amanda, Bob
Eli	Chris, Francis
Francis	Amanda, Bob, Chris, Eli

What is the fewest number of groups Dr. James can make and which students are grouped together?

Short Answer

7) Sierra is preparing a dinner for her friends. The following table shows the different steps involved in preparing and serving the dinner.

Tasks	Time (min.)	Prerequisite
A. Mix brownies	15 min.	none
B. Bake brownies	45 min.	A
C. Chop the vegetables	10 min.	none
D. Season the vegetables	5 min.	C
E. Prepare the chicken	15 min.	none
F. Cook the chicken	90 min.	E
G. Peel the potatoes	10 min.	none
H. Boil the potatoes	15 min.	G
I. Mash the potatoes	5 min.	H
J. Remove the chicken from oven to cool	10 min.	F
K. Roast the vegetables	15 min.	D, J
L. Serve the meal	None	B, I, K

What is the minimum amount of time Sierra needs to prepare and serve the dinner?

Long Answer

8) A company has taken on a new project and the table below shows the steps necessary to complete this project.

Tasks	Time (days)	Prerequisite
A.	4	none
B.	5	A
C.	6	A
D.	4	B
E.	10	B
F.	3	D
G.	5	C
H.	10	E, F, G
I.	none	H

The company has enough money to hire one extra person to help with these project.

Option 1: Hire someone to help with task C, which would reduce the time it takes to complete task C by 5 days

Option 2: Hire someone to help with task E, which would reduce the time it takes to complete task E by 4 days

Given the company wants to complete this project in as little time as possible, should they choose option 1 or option 2, and how much time will be save?