

**Math Workbook**

**Grade 11 CST**

**Answer Key**

**Term 3**

Population and Financial Math Unit – Exponents Review

An exponent is a number (or variable) that tells us how many times to multiply the base by itself.

$$y = a^x$$

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3$$

$$x^5 = x \times x \times x \times x \times x$$

**Laws of Exponents**

Name	Law	Example
Product	$a^m \cdot a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$
Quotient	$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$	$3^5 \div 3^2 = \frac{3^5}{3^2} = 3^{5-2} = 3^3$
Power	$(a^m)^n = a^{m \cdot n}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$
Negative Exponent	$a^{-m} = \frac{1}{a^m}$	$3^{-5} = \frac{1}{3^5}$
Zero Exponent	$a^0 = 1$	$3^0 = 1$
Fractional Exponent	$a^{m/n} = \sqrt[n]{a^m}$	$3^{2/5} = \sqrt[5]{3^2}$
Power of a Product	$(ab)^m = a^m \cdot b^m$	$(3 \cdot 4)^5 = 3^5 \cdot 4^5$
Power of a Fraction	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5}$

Population and Financial Math Unit – Exponents Review  
Practice Questions

1) Calculate each of the following expressions.

a)  $8^3$

b)  $5^{-3}$

c)  $16^{1/2}$

d)  $\sqrt[3]{125}$

e)  $\left(\frac{1}{2}\right)^{-1}$

f)  $135^0$

g)  $0^{13}$

h)  $1.025^1$

2) Determine if each of the following statements is true or false.

a)  $a^2 \times a^7 = a^9$

b)  $(a^3)^3 = a^6$

c)  $(a^2a^9)^3 = a^{33}$

d)  $\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}$

e)  $\frac{1}{a^{-2}} = a^{1/2}$

f)  $a^{3/7} = \sqrt[7]{a^3}$

## Population and Financial Math Unit – Exponents Review

If you are trying to solve for the base, the exponent becomes the root on the other side.

Ex: Solve for x in each of the following:

a)  $x^3 = 343$

$$x = \sqrt[3]{343}$$

$$x = 7$$

b)  $x^7 = 279\,936$

$$x = \sqrt[7]{279\,936}$$

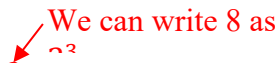
$$x = 6$$

If you are trying to solve for an exponent, re-write both sides with using the same base and then compare exponents.

(Note: We will learn a better way to solve these, using logarithms, but for now this is the approach we will use.)

Ex: Solve for x in each of the following

a)  $2^x = 8$

$2^x = 2^3$  

$$x = 3$$

b)  $729^x = 3$

$$(3^6)^x = 3$$

$$3^{6x} = 3^1$$

$$6x = 1$$

$$x = 0.167$$

### Practice Questions

3) Solve for x in each of the following.

a)  $x^3 = 8$

b)  $x^2 = \frac{1}{9}$

c)  $3^x = 81$

d)  $8^x = 2$

e)  $x^4 = 1296$

f)  $8^x = 64$

g)  $\left(\frac{1}{16}\right)^x = \frac{1}{4}$

$$2^x = \frac{1}{16}$$

## Population and Financial Math Unit – Population Questions

An exponential function is a function where  $x$  is the exponent.

$$y = x^3$$

$$y = 3^x$$

In general, we can write:  $y = ac^x$

Where:  $a$  is the initial value (starting amount)

$y$  is the final value (ending amount)

$c$  is the rate of change (growth or decay)

$x$  is the time

\*We always have to make sure that time is measured in the same units as the rate of growth/decay.

In a word problem, we can determine  $c$  as follows:

If we are given words: “doubles”,  $c = 2$ . “triples”,  $c = 3$ . “half”,  $c = 1/2$  (or 0.5), etc.

If we are given a percent:

$$\text{If growth: } c = 1 + \frac{\%}{100}$$

$$\text{If decay: } c = 1 - \frac{\%}{100}$$

Ex: An initial population of 1000 bacteria increases at a rate of 3% per day. What is the population after 4 days?

$$a = 1000$$

$$y = ?$$

$$c = 1 + \frac{3}{100} = 1.03 \quad (\text{measured in days})$$

$$x = 4 \quad (\text{measured in days})$$

$$y = ac^x$$

$$y = 1000 \times 1.03^4$$

$$y = 1000 \times 1.12551$$

$$y = 1125.51$$

Remember BEDMAS!  
Exponents before multiplication!

Therefore, the population of bacteria will be 1125

Ex: An initial population of 800 fish decreases by half every 6 months. What will the population be in 4 years?

$$a = 800$$

$$y = ?$$

$$c = 0.5 \quad (\text{measured in 6 months})$$

$$x = 4 \quad (\text{measured in years})$$

How many 6 month intervals are there in 4 years? 8.

$$\text{So } x = 8$$

$$y = ac^x$$

$$y = 800 \times 0.5^8$$

$$y = 800 \times 0.00391$$

$$y = 3.125$$

Therefore, there will be 3 fish.

## Population and Financial Math Unit – Population Questions

Ex: A city has a population of 30 000 people. The population is declining at a rate of 12% per year. What will the population be after 7 years?

$$a = 30\,000$$

$$y = ?$$

$$c = 1 - \frac{12}{100} = 0.88 \quad (\text{measured in years})$$

$$x = 7 \quad (\text{measured in years})$$

$$y = ac^x$$

$$y = 30\,000 \times 0.88^7$$

$$y = 30\,000 \times 0.40867559$$

$$y = 12\,260.2679$$

Therefore, the population will be 12 260

## Practice Questions

4) There are 140 residents in an apartment complex. It is expected that this number will increase at a rate of 0.5% every year. What will the population be after 5 years?

5) In an aquatic environment, there are 480 different species. It is expected that the number of species will decrease by 10% every year. How many species will there be after 8 years?

6) There are 4 bacteria in a petri dish. The population doubles every 20 minutes. What will the population be after 4 hours?

Population and Financial Math Unit – Population Questions

If we need to solve for either a or c, we can plug in the variables we know and use algebra to solve, or we can use the following formulas.

$$a = \frac{y}{c^x} \quad \text{and} \quad c = \sqrt[x]{\frac{y}{a}}$$

Ex: It is expected that the value of a stock will triple every 6 months. After 1 year, the value of the stock is \$10.80. What was the initial value of the stock?

$a = ?$	$y = ac^x$	$a = \frac{y}{c^x}$
$y = 10.8$		
$c = 3$ (measured 6 month intervals)	$10.8 = a \times 3^2$	
$x = 2$ (measured in 6 month intervals)	$10.8 = a \times 9$	O
	$a = 1.2$	$a = \frac{10.8}{3^2}$
		$a = \frac{10.8}{9}$
		$a = 1.2$

Therefore, the initial value is \$1.20

Ex: A grocery basket costs \$240 in 2018. If the cost of the same basket will be \$269.97 in 2021, what was the annual inflation rate during this time?

$a = 240$	$y = ac^x$	$c = \sqrt[x]{\frac{y}{a}}$
$y = 269.97$		
$c = ?$	$269.97 = 240 \times c^3$	
$x = 3$ (years)	$1.124875 = c^3$	O
	$c = 1.04$	$c = \sqrt[3]{\frac{269.97}{240}}$
		$c = \sqrt[3]{1.124875}$
		$c = 1.04$

Recall  $c = 1 + \frac{\%}{100}$

$$1.04 = 1 + \frac{\%}{100}$$

$$0.04 = \frac{\%}{100}$$

$$4 = \%$$

Therefore, the inflation rate was 4%

## Population and Financial Math Unit – Population Questions

### Practice Questions

7) A student posts a photograph online. On the first day the photo is posted, it is seen by 38 people. After 10 days, the photo has been seen by 2191. What is the rate by which the number of people seeing the photo increases each day?

8) Calcium chloride is used to de-ice roads during Quebec's winters. During an ice storm, 7.5 cm of ice accumulates on the roads because the de-icer spreads the salt. After 3 hours the thickness of the ice will be 5.11 cm? By what rate does the salt reduce the thickness of ice per hour?

9) Super Balls are made out of a synthetic rubber. They bounce much higher than other balls. At each bounce, a Super Ball loses 8% of its maximum height from the previous bounce. After 8 bounces, the ball has a height of 200 cm. From what height was the ball originally dropped?



## Population and Financial Math Unit – Logarithm Basics

We have solved exponential functions for every variable except  $x$ . In order to solve for  $x$ , we need to use logarithms.

A logarithm is a function (and a button on your calculator, just like the trig functions, sine, cosine, and tangent).

We can re-write an exponential function as a logarithmic function.

$$y = c^x \Leftrightarrow x = \log_c y$$

Ex: Write the following expressing using logs

a)  $4 = 5^x$

$$x = \log_5 4$$

b)  $150 = 3^x$

$$x = \log_3 150$$

c)  $6561 = 3^8$

$$8 = \log_3 6561$$

Some calculators can only calculate logs with a base on 10 (where  $c = 10$ ) and others allow you to change the base. If your calculator and you will have to use the following:

$$\log_c y = \frac{\log y \text{ (log of the big number)}}{\log c \text{ (log of the little number)}}$$

Ex: Solve for  $x$

$$8 = 2^x$$

$$x = \log_2 8$$

$$x = \frac{\log 8}{\log 2}$$

$$x = 3$$

### Practice Questions

10) Write each of the following exponential equations as logs

a)  $8 = 16^x$

b)  $0.58 = 1.2^x$

c)  $5.37824 = 1.4^5$

11) Solve each of the following for  $x$ , using logs

a)  $20 = 3^x$

b)  $1.79 = 2.5^x$

c)  $120.5 = 1.03^x$

## Population and Financial Math Unit – Using Logs to Solve Population Problems

In order to solve an exponential function for  $x$  you can either plug in what you know and use algebra (remembering to solve using logs) or you can use the following formula:

$$x = \frac{\log\left(\frac{y}{a}\right)}{\log(c)}$$

Ex: Bill has 2g of cells in a petri dish. The amount of cells increases by 12% every month. How long does it take for Bill to have 4g of cells?

$$a = 2$$

$$y = 4$$

$$c = 1 + \frac{12}{100} = 1.12 \quad (\text{per}$$

month)

$$x = ? \text{ (months)}$$

$$y = ac^x$$

$$4 = 2 \times 1.12^x$$

$$2 = 1.12^x$$

$$x = \frac{\log 2}{\log 1.12}$$

$$x = 6.1$$

$$x = \frac{\log\left(\frac{y}{a}\right)}{\log(c)}$$

$$x = \frac{\log\left(\frac{4}{2}\right)}{\log(1.12)}$$

$$x = \frac{\log 2}{\log 1.12}$$

$$x = 6.1$$

Therefore, it will take 6.1 months

### Practice Questions

12) A petri dish contains 150 bacteria. The number of bacteria increases by 20% every hour. How long will it take for there to be 500 bacteria?

13) Water from a tank evaporates at a rate of 1% of its volume every hour. After how many hours will the tank hold 4184 L of water, if the tank initially contained 4650 L of water?

## Population and Financial Math Unit – Simple Interest

When working with financial math (ex: finding the future value of an investment, etc.), the question may ask about simple interest or compound interest.

Simple interest means you only earn interest on your initial investment. For example, if you invest \$500 at an annual interest rate of 1%, you would earn 1% of \$500 (or \$5) for each year you kept your money in the investment.

We do not need exponential functions or logs to calculate investments with simple interest, but instead use the formula:

$$C(t) = C_0(1 + it)$$

Where:

$C(t)$  is the final amount

$C_0$  is the initial investment

$i$  is the interest rate  $\left(\frac{\%}{100}\right)$

$t$  is the time (if necessary, transform  $t$  so it is in the same units as the interest rate)

Ex: Jill invests \$5000 at a simple interest rate of 6% annually. How much money does Jill have after 5 years?

$$C(t) = ?$$

$$C(t) = C_0(1 + it)$$

$$C_0 = 5000$$

$$C(t) = 5000(1 + 0.06 \times 5)$$

$$i = \frac{6}{100} = 0.06$$

$$C(t) = 5000(1 + 0.3)$$

$$t = 5$$

$$C(t) = 5000(1.3)$$

$$C(t) = 6500$$

Jill has \$6500

Remember BEDMAS!  
Brackets first  
and  
Multiplication before addition!

Ex: Ryan invests \$1000 at an annual simple interest rate of 4%. How much money does Ryan have after 18 months?

$$C(t) = ?$$

$$C(t) = C_0(1 + it)$$

$$C_0 = 1000$$

$$C(t) = 1000(1 + 0.04 \times 1.5)$$

$$i = \frac{4}{100} = 0.04$$

$$C(t) = 1000(1 + 0.06)$$

$$t = 1.5$$

$$C(t) = 1000(1.06)$$

(18 months written in terms of years)

$$C(t) = 1060$$

Ryan has \$1060

## Population and Financial Math Unit – Simple Interest

### Practice Questions

14) Determine the accumulated capital (the final amount of the investment) if you invested \$8000 for 5 years at an annual simple interest rate of 7%.

15) Carlos invests \$3600 at a monthly simple interest rate of 0.45%. What is the value of Carlos' investment after 7 years?

16) Mikka is saving money to buy a car. She wants to have at least \$5000 for the down payment and so she makes two investments.

Investment 1: Mikka invests \$2247.19 at a monthly simple interest rate of 1.3% for 3 years.

Investment 2: Mikka invests \$1200 at a weekly simple interest rate of 0.3% for 3 years.

After 3 years, will Mikka have reached her goal?

## Population and Financial Math Unit – Simple Interest

We will not always be asked to solve for the accumulated capital (final value) of an investment. We can solve for the other variables by plugging in what we know and using algebra to solve, or by using the following formulas:

$$C_0 = \frac{C(t)}{1 + it}$$

$$t = \frac{\frac{C(t)}{C_0} - 1}{i}$$

$$i = \frac{\frac{C(t)}{C_0} - 1}{t}$$

Ex: Rasha invested some money for 11 years at an annual simple interest rate of 7.4%. After 11 years, the investment was worth \$7890.90. How much did Rasha invest initially?

$$C(t) = 7890.90$$

$$C(t) = C_0(1 + it)$$

$$C_0 = \frac{C(t)}{1 + it}$$

$$C_0 = ?$$

$$7890.90 = C_0(1 + 0.074 \times 11)$$

$$C_0 = \frac{7890.90}{1 + 0.074 \times 11}$$

$$i = \frac{7.4}{100} \\ = 0.074$$

$$7890.90 = C_0(1 + 0.814)$$

$$C_0 = \frac{7890.90}{1 + 0.814}$$

$$7890.90 = C_0(1.814)$$

$$C_0 = \frac{7890.90}{1.814}$$

$$t = 11$$

$$C_0 = 4350$$

$$C_0 = 4350$$

Rasha invested \$4350

Ex: How long will it take an initial capital of \$9300 to earn an accumulated capital of \$14 322 at an annual simple interest rate of 8%?

$$C(t) = 14\,322$$

$$C(t) = C_0(1 + it)$$

$$t = \frac{\frac{C(t)}{C_0} - 1}{i}$$

$$C_0 = 9300$$

$$14\,322 = 9300(1 + 0.08 \times t)$$

$$t = \frac{\frac{14\,322}{9300} - 1}{0.08}$$

$$i = \frac{8}{100} = 0.08$$

$$1.54 = 1 + 0.08 \times t$$

$$0.54 = 0.08 \times t$$

$$t = \frac{1.54 - 1}{0.08}$$

$$t = ?$$

$$t = 6.75$$

$$t = \frac{0.54}{0.08}$$

$$t = 6.75$$

It will take 6.75 years

## Population and Financial Math Unit – Simple Interest

Ex: What is the annual simple interest rate for an investment of \$2400 that yields \$4646.40 after 9 years.

$$C(t) = 4646.40$$

$$C(t) = C_0(1 + it)$$

$$i = \frac{\frac{C(t)}{C_0} - 1}{t}$$

$$C_0 = 2400$$

$$4646.40 = 2400(1 + i \times 9)$$

$$i = ?$$

$$1.936 = 1 + i \times 9$$

$$i = \frac{\frac{4646.40}{2400} - 1}{9}$$

$$t = 9$$

$$0.936 = i \times 9$$

$$i = \frac{1.936 - 1}{9}$$

$$i = 0.104$$

$$i = \frac{0.936}{9}$$

$$\text{Recall } i = \frac{\%}{100}$$

$$t = 0.104$$

$$0.104 = \frac{\%}{100}$$

$$10.4 = \%$$

Therefore, the annual simple interest rate was  
10.4%

### Practice Questions

17) After 7.5 years, an investment is worth \$4199.75. What was the amount of the initial investment given that the simple interest rate was 3.8% per half-year?

18) \$1800 is invested at a monthly simple interest rate of 2.25%. How long does it take for the investment to be worth \$3420?

## Population and Financial Math Unit – Simple Interest

19) \$9000 is invested for 7 years and 4 months. It is now worth \$18 504. What was the monthly simple interest rate?

20) You borrowed \$3700 at a daily simple interest rate of 0.02%. When you repaid the loan, you paid \$4780.40. How long was it before you repaid the loan?

21) After 2.5 years, the repayment of a debt will be \$11 770.75 at a weekly simple interest rate of 0.15%. How much money was borrowed initially?

22) The repayment of a \$7 350 loan is \$14 891.10 after 9 years. Determine the quarterly simple interest rate.

## Population and Financial Math Unit – Compound Interest

In simple interest, you only earn interest on your initial investment.

In compound interest, the interest you have previously earned is added to your initial investment before the next round of interest is calculated. In other words, interest is earned on interest.

To calculate compound interest, we use the following formula:

$$C(t) = C_0(1 + i)^t$$

Where:

$C(t)$  is the final amount

$C_0$  is the initial investment

$i$  is the interest rate  $\left(\frac{\%}{100}\right)$

$t$  is the time (if necessary, transform  $t$  so it is in the same units as the interest rate)

\*Note: this is just like the exponential functions we were looking at earlier in this unit, but using different letters.

Also, if a question asks about depreciation, you will subtract  $i$ , using this formula  $C(t) = C_0(1 - i)^t$

Ex: Miguel invests \$500 at an annual compound interest rate of 2.6%. How much is Miguel's investment worth after 8 years?

$$C(t) = ?$$

$$C(t) = C_0(1 + i)^t$$

$$C_0 = 500$$

$$C(t) = 500(1 + 0.026)^8$$

$$i = \frac{2.6}{100} = 0.026$$

$$C(t) = 500(1.026)^8$$

$$t = 8$$

$$C(t) = 500 \times 1.2279449$$

$$C(t) = 613.97$$

Miguel will have \$613.97

Ex: You borrow \$700 at a monthly compound interest rate of 1.5%. How much will you have to repay after 2 years?

$$C(t) = ?$$

$$C(t) = C_0(1 + i)^t$$

$$C_0 = 700$$

$$C(t) = 700(1 + 0.015)^{24}$$

$$i = \frac{1.5}{100} = 0.015$$

$$C(t) = 700(1.015)^{24}$$

$$t = 24$$

$$C(t) = 700 \times 1.429502812$$

$$C(t) = 1000.65$$

You will have to repay \$1000.65



Population and Financial Math Unit – Compound Interest  
Practice Questions

23) An amount of \$5000 is invested at an interest rate of 6% compounded annually. What is the value of this investment in 5 years?

24) You purchase a new computer for \$1200. The value of the computer depreciates at a rate of 43% annually. How much will your computer be worth after 3 years?

25) Valerie invests \$2500 at an interest rate of 2.3% compounded quarterly. What is the value of her investment after 6 years?

Population and Financial Math Unit – Compound Interest

Just like in simple interest, we will not always be asked to solve for the accumulated capital (final value) of an investment.

We can solve for the other variables by plugging in what we know and using algebra to solve, or by using the following formulas:

$$C_0 = \frac{C(t)}{(1+i)^t}$$

$$i = \sqrt[t]{\frac{C(t)}{C_0}} - 1$$

Ex: After 8 years, the accumulated capital of an investment at an annual compound interest rate of 2% will be \$7029.96. What was the amount of the initial investment?

$$C(t) = 7029.96$$

$$7029.96 = C_0(1 + 0.02)^8$$

$$C_0 = \frac{C(t)}{(1+i)^t}$$

$$C_0 = ?$$

$$7029.96 = C_0(1.02)^8$$

$$i = \frac{2}{100} = 0.02$$

$$7029.96 = C_0 \times 1.171659381$$

OR

$$C_0 = \frac{7029.96}{(1 + 0.02)^8}$$

$$t = 8$$

$$C_0 = 6000.00$$

$$C_0 = \frac{7029.96}{(1.02)^8}$$

$$C_0 = \frac{7029.96}{1.171659381}$$

$$C_0 = 6000.00$$

The initial investment was \$6000.00

Ex: \$4700 was invested and yields \$8699.37 after 8 years. What was the annual compound interest rate?

$$C(t) = 8699.37$$

$$8699.37 = 4700(1 + i)^8$$

$$i = \sqrt[t]{\frac{C(t)}{C_0}} - 1$$

$$C_0 = 4700$$

$$1.850929787 = (1 + i)^8$$

OR

$$i = \sqrt[8]{\frac{8699.37}{4700}} - 1$$

$$i = ?$$

$$1.08 = 1 + i$$

$$t = 8$$

$$0.08 = i$$

$$i = \sqrt[8]{1.850929787} - 1$$

$$\begin{aligned} \text{Recall } i &= \frac{\%}{100} \\ 0.08 &= \frac{\%}{100} \\ 8 &= \% \end{aligned}$$

$$i = 1.08 - 1$$

$$i = 0.08$$

The annual compound interest rate was 8%

## Population and Financial Math Unit – Compound Interest

To solve for  $t$ , we need to use logs, like we did with population questions. You can either plug in what you know and use algebra, or you can use this formula:

$$t = \frac{\log\left(\frac{C(t)}{C_0}\right)}{\log(1 + i)}$$

Ex: An initial investment of \$3000 earns an annual compound interest rate of 2%. How long will it take for the investment to be worth \$3312.24?

$$C(t) = 3312.24$$

$$3312.24 = 3000(1 + 0.02)^t$$

$$C_0 = 3000$$

$$1.10408 = (1.02)^t$$

$$t = \frac{\log\left(\frac{C(t)}{C_0}\right)}{\log(1 + i)}$$

$$i = \frac{2}{100} = 0.02$$

$$t = \frac{\log 1.10408}{\log 1.02}$$

$$t = \frac{\log\left(\frac{3312.24}{3000}\right)}{\log(1 + 0.02)}$$

$$t = ?$$

$$t = 5$$

$$t = \frac{\log(1.10408)}{\log(1.02)}$$

$$t = 5$$

It will take 5 years.

### Practice Questions

26) Money is invested at an interest rate of 4% compounded annually. If the future value of this investment after 6 years is \$3796, what is the value of the initial investment?

27) \$2500 is invested at an annual compound interest rate of 3%. How long will it take for the investment to be worth \$2985?

Population and Financial Math Unit – Compound Interest

28) \$2800 was invested and after 2 years the value of the investment was \$3087. What was the annual compound interest rate?

29) \$5000 is invested at an annual interest rate of 2% compounded annually. How long will it take for the investment to be worth \$5520.40?

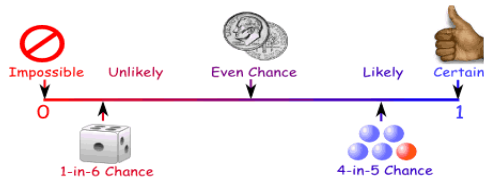
30) At what annual compound interest rate should we invest so that our initial investment will be doubled after 10 years?

31) In 8 years, the accumulated capital of an investment at an annual compound interest rate of 2% will be \$7029.96. What was the value of the initial investment?

Probability Unit – Review

**REVIEW**

Probability is the chance something will happen. This can be shown on a probability line.



We can also use numbers (such as fractions or decimals) to show the probability of something happening.

- Impossible is a zero
- Certain is a one

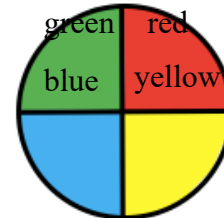
We can calculate the probability of an event using the following formula:

<b>Probability of Event A: P(A)</b>
$P(A) = \frac{\text{The number of ways event A can occur}}{\text{The total number of possible outcomes}}$

Ex: A spinner has 4 equal sectors coloured yellow, blue, green and red.

What are the chances of landing on blue?  $P(\text{blue}) = \frac{1}{4}$

What are the chances of landing on red?  $P(\text{red}) = \frac{1}{4}$



Ex: Where would the following statements sit on the probability line?

- Choosing a green ball from a bag with only 4 green balls
- Choosing a green ball from a bag with only 2 red balls and 4 yellow balls
- Choosing a green ball from a bag with 2 green balls and 2 red balls
- Choosing a red ball from a bag with 1 red ball and 3 green balls



Probability Unit – Review  
Practice Questions

1) A jar contains 100 jelly beans. There are 20 red jelly beans, 10 black jelly beans, 15 orange jelly beans, 30 green jelly beans, and 25 yellow jelly beans. If you chose a single jelly bean at random, what is the probability that you would choose an orange jelly bean?

2) In a standard deck of 52 playing cards, what is the probability of randomly drawing:

a) a queen?

b) a red card?

c) a club?

d) the 7 of diamonds?

3) A bag contains 4 red marbles, 2 green marbles, 6 white marbles, and 3 blue marbles. What is the probability of randomly drawing:

a) a red marble?

b) a green marble?

c) a white marble?

d) a blue marble?

**Definitions**

<b>Term</b>	<b>Definition</b>	<b>Example</b>
Experiment	An <b>experiment</b> is a situation involving chance or probability that leads to results called outcomes.	In the spinner question above, the experiment is spinning the spinner.
Outcome	An <b>outcome</b> is the result of a single trial of an experiment.	The possible outcomes are landing on yellow, blue, green or red.
Sample Space	The <b>sample space</b> is the list of all possible outcomes. Notation is ‘omega’ $\Omega$	$\Omega = \{\text{yellow, blue, green, red}\}$
Random	An experiment is <b>random</b> is the outcome depends on chance (the outcome cannot be predicted with absolute certainty) and we can list the sample space before the experiment.	The spinner example is a random experiment because we cannot predict the outcome with certainty and we know the sample space.
Event	An <b>event</b> is one or more outcomes of an experiment. This is also known as a <b>subset</b> of the sample space.	One event of this experiment would be landing on blue.
Simple Event	A <b>simple event</b> has only one outcome.	Ex: If the spinner was all blue
Probability	<b>Probability</b> is the measure of how likely an event is.	The probability of landing on blue is $\frac{1}{4}$ .
Theoretical Probability	The <b>theoretical probability</b> of an event is a number that quantifies the possibility that the event will occur – it is determined by mathematical reasoning. <i>Theoretical Probability</i> $= \frac{\# \text{ of favorable outcomes}}{\# \text{ of possible outcomes}}$	For example, in the spinner question above, the theoretical probability of landing on green can be calculated and is $\frac{1}{4}$ .
Experimental Probability	The <b>experimental probability</b> of an event is determined from the experiment itself. <i>Experimental Probability</i> $= \frac{\# \text{ of times outcomes occurs}}{\# \text{ of trials}}$	If we spun the spinner 10 times and landed on green 3 times, red 2 times, blue 2 times, and yellow 3 times, the subjective probability of landing on green is $\frac{3}{10}$ .
Subjective Probability	The <b>subjective probability</b> that an event will occur relies on judgment or perceptiveness of a person who has a particular set of information about the situation	An example of subjective probability is predicting the weather. Ex. Someone says there is a 60% chance of rain tomorrow.

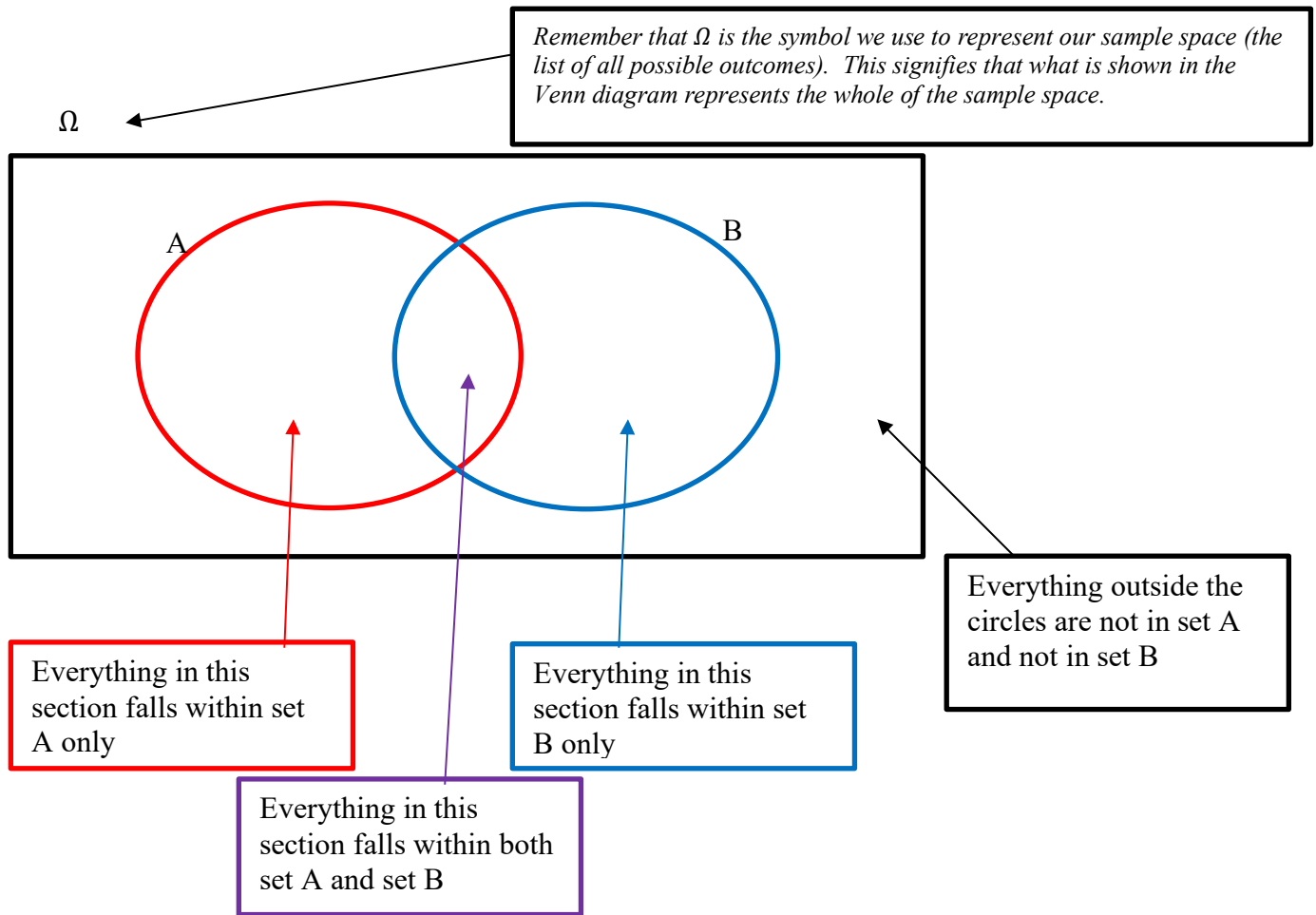
Probability Unit – Logical Connectors and Venn Diagrams

**LOGICAL CONNECTORS AND VENN DIAGRAMS**

The logical connectors “and” and “or” can be used to describe an event.

- AND: The intersection of two sets A and B, written  $A \cap B$  is composed of elements that are common to both sets.
- OR: The union of two sets A and B, written  $A \cup B$  is composed of all elements in either A, B, or both.

We can use a Venn diagram to represent these possibilities.

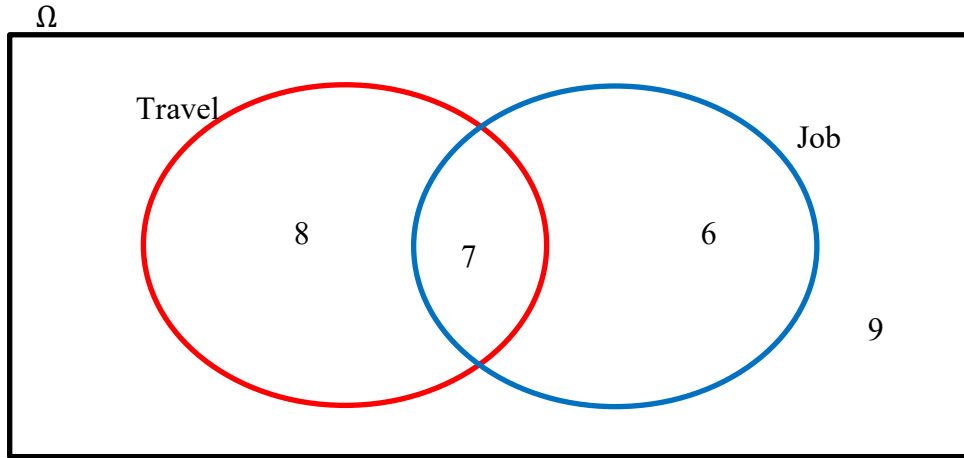




Probability Unit – Logical Connectors and Venn Diagrams

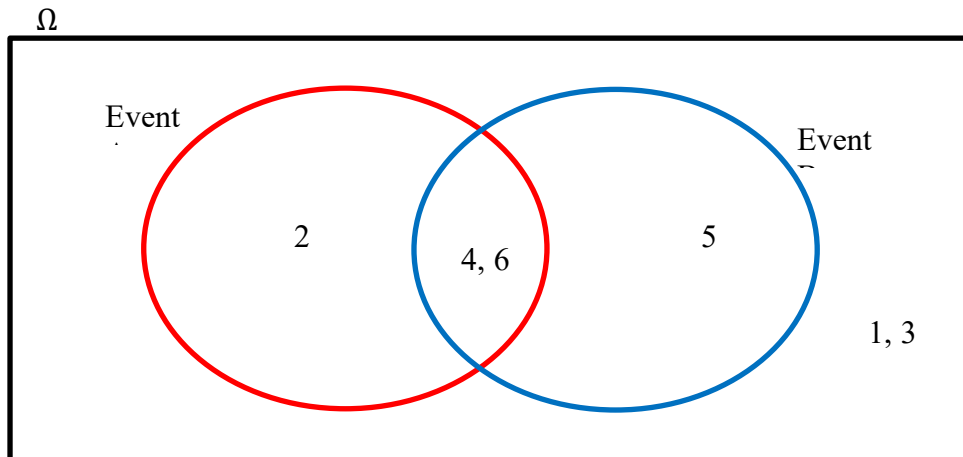
Ex: Dr. James asked 30 students the following questions: 1) Do you travel more than 30 minutes to get to school? 2) Do you have a job? The results of the survey are presented in the table below and depicted in the Venn Diagram.

Yes to Q1 only (travel)	Yes to Q2 only (job)	Yes to both Q1 and Q2	No to both
8 students	6 students	7 students	9 students



Ex: You are rolling a 6-sided die (where the faces are numbered from 1-6). Create a Venn diagram to represent the Events A and B below.

- Event A: Rolling an even number Set A is composed of three possibilities: Rolling a 2, 4, or 6
- Event B: Rolling a number larger than 3 Set B is composed of three possibilities: Rolling a 4, 5, or 6
- $A \cap B$  corresponds to the numbers in both Event A and Event B (even numbers larger AND larger than 3)  $A \cap B$  is composed of two possibilities: 4 and 6
- $A \cup B$  corresponds to all the numbers that are in Event A OR Event B (or both)  $A \cup B$  is composed of four possibilities: 2, 4, 5 and 6

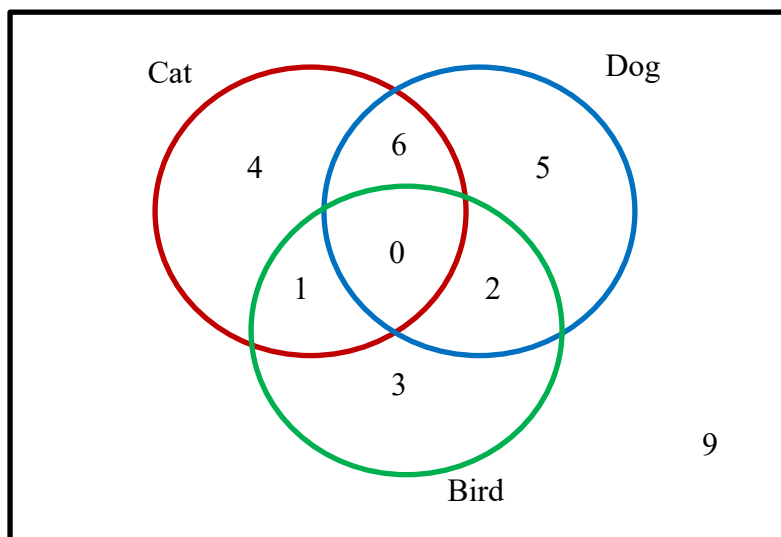


Probability Unit – Logical Connectors and Venn Diagrams

We can also create Venn Diagrams with more than two circles.

Ex: Dr. James asked 30 students the following questions: 1) Do you own a cat? 2) Do you own a dog? 3) Do you own a bird? The results of the survey are presented in the table below and depicted in the Venn Diagram.

Yes to Cat only	Yes to Dog only	Yes to Bird only	Yes to Cat and Dog	Yes to Cat and Bird	Yes to Dog and Bird	Yes to Cat, Bird, and Dog	No to all
4 students	5 students	3 students	6 students	1 student	2 students	0	9 students



Practice Questions

4) There are 10 friends, Alex, Blaire, Casey, Drew, Erin, Francis, Glen, Hunter, Ira, and Jade.  
 Alex, Casey, Drew, and Hunter all play soccer.  
 Casey, Drew, and Jade all play tennis.  
 Create a Venn diagram to represent this scenario.

5) There are 10 friends, Alex, Blaire, Casey, Drew, Erin, Francis, Glen, Hunter, Ira, and Jade.  
 Alex, Casey, Drew, and Hunter all play soccer.  
 Casey, Drew, and Jade all play tennis.  
 Drew, Glen, and Jade all play volleyball.  
 Create a Venn diagram to represent this scenario.

Probability Unit – Logical Connectors and Venn Diagrams

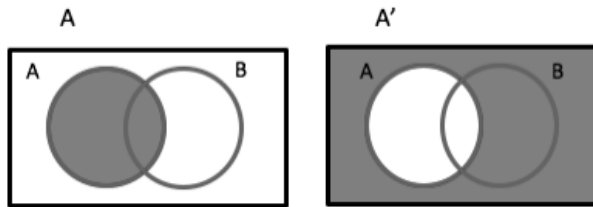
**LOGICAL CONNECTORS REPRESENTED VISUALLY**

The **complement** of an event is the collection of outcomes that are NOT the event.

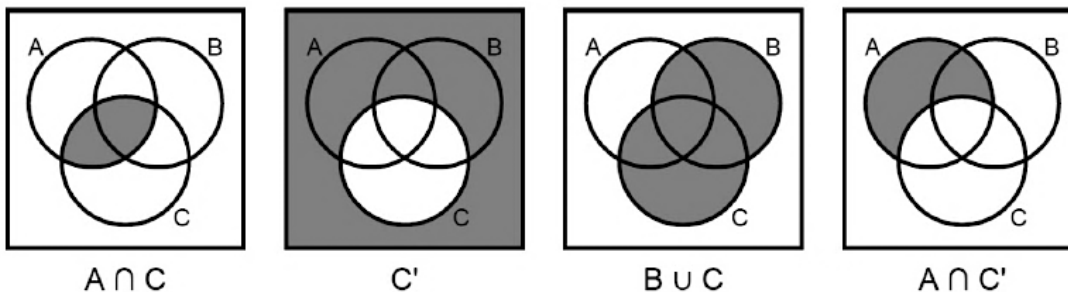
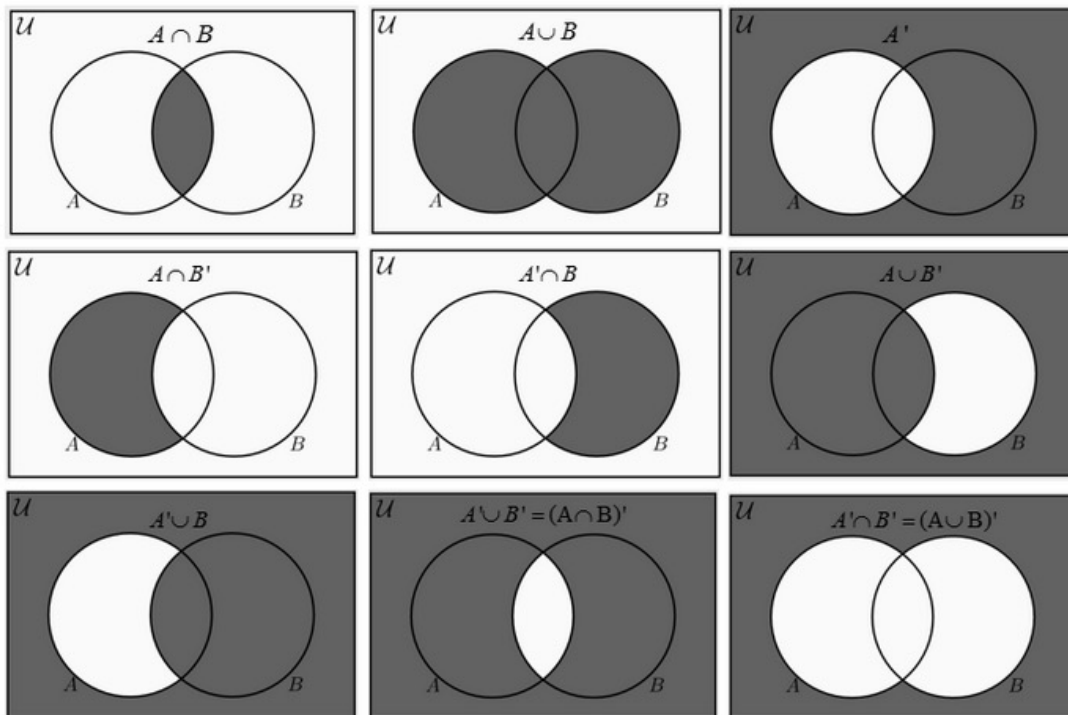
In the spinner example, if the event (A) is spinning the spinner and landing on BLUE, the complementary event (A') is landing on NOT BLUE (everything other than blue).

When you add the probability of an event, (A) to the probability of the complement of the event (A'), the result will always be 1. It is sometimes easier to determine P(A) using this method:

$$P(A) + P(A') = 1$$



Here is a visual representation of many different statements with logical connectors:



## Probability Unit – Logical Connectors and Venn Diagrams

Ex: You roll two dice (6-sided). What is the probability that the number that lands face up is different on the two dice.

Let  $A = \text{landing on two different numbers}$ . So  $A' = \text{landing on the same number}$ .

$$\Omega_{A'} = \{1 - 1, 2 - 2, 3 - 3, 4 - 4, 5 - 5, 6 - 6\}$$

Given that you are rolling 2 dice, each with 6 options, there are a total 36 possible combinations.

$$P(A) + P(A') = 1$$

$$P(A) + \frac{6}{36} = 1$$

$$P(A) + \frac{6}{36} = \frac{36}{36}$$

$$P(A) = \frac{36}{36} - \frac{6}{36}$$

$$P(A) = \frac{30}{36} = \frac{5}{6}$$

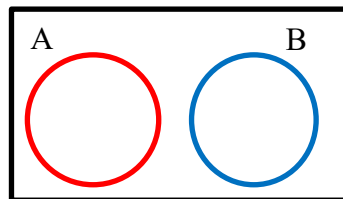
Therefore, the probability of landing on two different numbers is  $\frac{5}{6}$ .

Events are called **mutually exclusive** if they cannot happen at the same time.

Ex: If we are spinning a spinner one time and Event A=spinning BLUE and Event B=spinning RED, those events are mutually exclusive because you cannot land on BLUE and RED at the same time.

In a Venn Diagram, two mutually exclusive events do not overlap (or if the circles are drawn as overlapping, there is nothing in that overlapping section).

$$P(A \cap B) = 0$$



### Practice Questions

6) You are flipping a coin that has been weighted so that the chances it lands on tails is  $\frac{3}{5}$  (instead of  $\frac{1}{2}$ ). What is the probability that this coin will land on heads?

7) A 6-sided die with sides numbered 1 to 6 is rolled twice. The following are two possible events:

A: obtaining a sum that is even

B: obtaining a sum that is greater than or equal to 7

a) Are events A and B mutually exclusive? Explain.

b) Calculate the following

i)  $P(A)$

ii)  $P(B)$

## Probability Unit – Probability with Logical Connectors

### DETERMINING PROBABILITY OF EVENTS WITH LOGICAL CONNECTORS

We can determine the probability of events with AND and OR statements.

#### Probability of an event with an AND statement

$$P(A \cap B) = P(A) \times P(B)$$

Ex: You randomly draw a card from a standard deck of playing cards. What is the probability that you draw a red card and a queen?

*A = draw a red card*

*B = draw a queen*

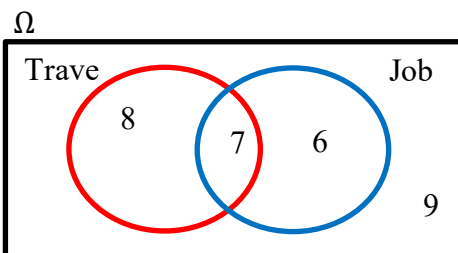
$$P(A \cap B) = P(A) \times P(B)$$

$$P(A \cap B) = \frac{1}{2} \times \frac{1}{13}$$

$$P(A \cap B) = \frac{1}{26}$$

We can also determine the probability of an AND statement from a Venn Diagram.

Ex: Dr. James asked 30 students the following questions: 1) Do you travel more than 30 minutes to get to school? 2) Do you have a job? The results are presented in the Venn Diagram below. What is the probability that a student travels more than 30 minutes to get to school AND has a job.



There are 7 students in the section where the circles overlap and there are 30 students total (8+7+6+9). Therefore:

$$P(T \cap J) = \frac{7}{30}$$

#### Probability of an event with an OR statement

To find the probability of  $A \cup B$ , we have to add the probabilities of A and B, and then subtract the overlapping area (otherwise it would be added twice)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex: You are rolling a 6-sided die one time. What is the probability of rolling a number larger than 3 or an even number?

*A = rolling a number larger than 4*

*B = rolling an even number*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

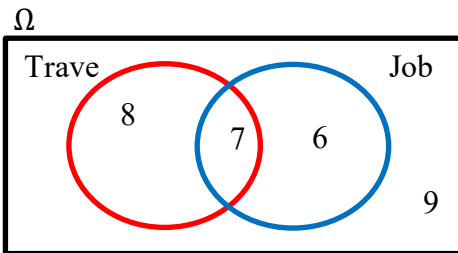
$$P(A \cup B) = \frac{2}{6} + \frac{3}{6} - \frac{1}{6}$$

$$P(A \cup B) = \frac{4}{6} = \frac{2}{3}$$

## Probability Unit – Probability with Logical Connectors

We can also determine the probability of an OR statement from a Venn Diagram.

Ex: Dr. James asked 30 students the following questions: 1) Do you travel more than 30 minutes to get to school? 2) Do you have a job? The results are presented in the Venn Diagram below. What is the probability that a student travels more than 30 minutes to get to school OR has a job.



There are 15 students who travel more than 30 minutes to get to school (8+7) and there are 13 students who have a job (6+7). Additionally, there are 7 students who both travel more than 30 minutes and have a job.

There are 30 students total (8+7+6+9). Therefore:

$$P(T \cap J) = \frac{15}{30} + \frac{13}{30} - \frac{7}{30}$$

$$P(T \cap J) = \frac{21}{30} = \frac{7}{10}$$

Note: When using a Venn Diagram, you can also simply count the number of elements in T only, J only, and where T and J overlap, and divide by the total number of elements:

$$P(T \cap J) = \frac{8 + 7 + 6}{30} = \frac{21}{30} = \frac{7}{10}$$

### Practice Questions

8) You are rolling a 6-sided die one time. What is the probability of rolling:

- a 3?
- an even number or a 5?
- a 3 or a 6?
- an even number that is also greater than 2?

9) Pool is played on a table with 16 balls. One of the balls is an unnumbered white ball. The balls numbered 1 to 8 are solid colours. The balls numbered 9 to 15 have a coloured stripe. If one of these balls is chosen at random, what is the probability of choosing:

- an even numbered ball?
- a solid coloured ball?
- an even numbered ball that is also solid-coloured?
- an odd numbered ball or a striped ball?

## Probability Unit – Completing a Venn Diagram

### Completing a Venn Diagram

Ex: 35 students were polled

- 16 students live on a farm
- 17 students have a sibling
- 28 students have a sibling or have a pet
- 3 students who live on a farm own a pet and have a sibling
- 5 students have a sibling and own a pet
- 11 students live on a farm and have a sibling
- 26 students own a pet or live on a farm

Step 1: Determine A, B, and C

A: Live on a farm

B: Have a sibling

C: Own a pet

Step 2: Turn statements into set builder notation (use A, B, C and  $\cup$  or  $\cap$ )

- 16 students live on a farm:  $A = 16$
- 17 students have a sibling  $B = 17$
- 28 students have a sibling or have a pet  $B \cup C = 28$
- 3 students who live on a farm own a pet and have a sibling  $A \cap B \cap C = 3$
- 5 students have a sibling and own a pet  $B \cap C = 5$
- 11 students live on a farm and have a sibling  $A \cap B = 11$
- 26 students own a pet or live on a farm  $A \cup C = 26$

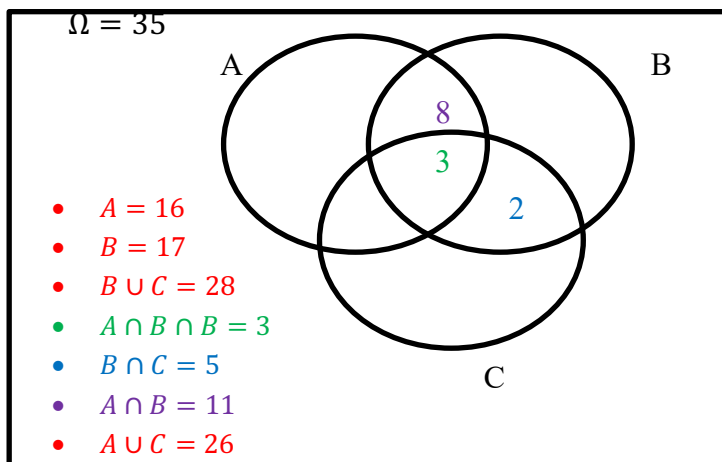
Step 3: Fill in Venn Diagram (start with the “ANDS”)

Since  $A \cap B \cap C = 3$ , we know the section where A, B, and C all overlap must be 3.

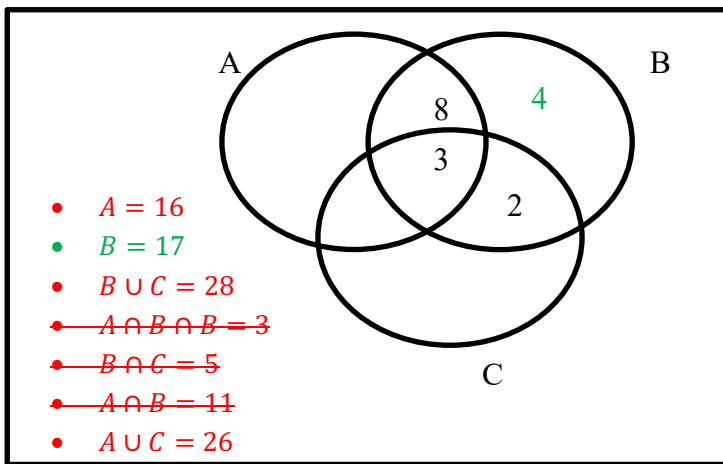
Since  $B \cap C = 5$ , we know the section where B and C overlap must add to 5. Note that there are 2 areas where B and C overlap – where B and C overlap only, and where A, B, and C all overlap. Since there are already 3 in the section where A, B, and C overlap, the section where only B and C overlap must be  $5 - 3 = 2$ .

Since  $A \cap B = 11$ , we know the section where A and B overlap must add to 11. Note that there are 2 areas where A and B overlap – where A and B overlap only, and where A, B, and C all overlap. Since there are already 3 in the section where A, B, and C overlap, the section where only A and B overlap must be

$$11 - 3 = 8$$



## Probability Unit – Completing a Venn Diagram



Now that we have all the ANDS done, move on to any areas that only have one section remaining to fill in. See how, in our example, A still have 2 sections remaining to fill in, as does C. But B only has one area left, so we should be able to figure out what belongs in there.

Since  $B = 17$ , that means that if we add all the sections that are part of B, we will get 17. So

$$8 + 3 + 2 + ? = 17$$

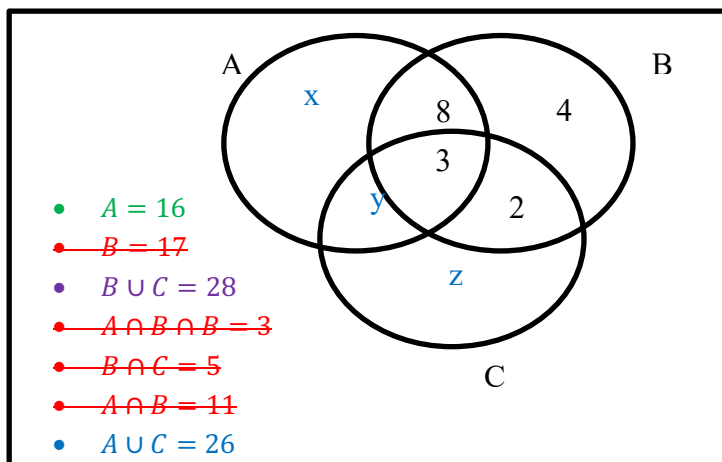
OR

$$17 - 8 - 3 - 2 = ?$$

$$4 = ?$$

Now we have to do some careful reasoning to determine the values in the remaining sections.

Let's give them some labels.



We know  $A = 16$  so the sum of all the areas that make up A must be 16. Therefore:

$$8 + 3 + x + y = 16$$

$$x + y = 5$$

We also  $B \cup C = 28$  so the sum of all the areas in B and C must total 28. Therefore:

$$8 + 4 + 3 + 2 + y + z = 28$$

$$y + z = 11$$

Similarly,  $A \cup C = 26$ , so the sum of all the areas in A and C must total 26. Therefore:

$$8 + 3 + 2 + x + y + z = 26$$

$$x + y + z = 13$$

In the 3 equations above ( $x + y = 5$ ,  $y + z = 11$ , and  $x + y + z = 13$ ), we have 3 variables. Because the number of equations and the number of variables match, we should be able to solve. Let's start with the equation that has all 3 variables.

$$x + y + z = 13$$

We can then choose either other equation and use substitution. Let's use  $x + y = 5$

$$x + y + z = 13$$

$$5 + z = 13$$

Now we can solve for z.

$$5 + z = 13$$

$$z = 8$$

Now that we know the value of z, we can pick the equation with 2 variables that includes z

$$y + z = 11$$

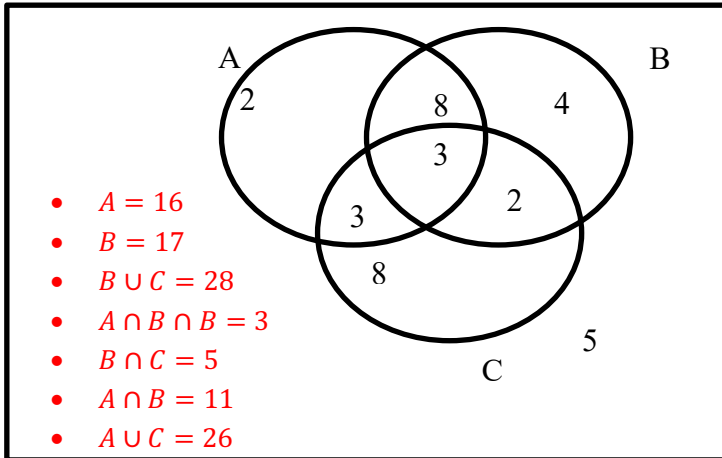
$$y + 8 = 11$$

$$y = 3$$



Probability Unit – Completing a Venn Diagram

$$\Omega = 35$$



Finally, we can use any equation that includes  $x$  to solve for  $x$

$$x + y = 5$$

$$x + 3 = 5$$

$$x = 2$$

And now we can use those values to complete the Venn Diagram.

We should check to see if all students are accounted for:

$$2 + 8 + 4 + 3 + 3 + 2 + 8 = 30$$

But there were 35 students, so 5 must be outside the circles.

Using the Venn Diagram, we can now determine any probabilities, for example:

$$P(C) = \frac{8 + 3 + 3 + 2}{35} = \frac{16}{35}$$

$$P(A \cap B) = \frac{8 + 3}{35} = \frac{11}{35}$$

$$P(A \cup B) = \frac{2 + 3 + 8 + 3 + 4 + 2}{35} = \frac{22}{35}$$

$$P(A \cup B \cup C) = \frac{2 + 8 + 3 + 3 + 4 + 2 + 8}{35} = \frac{30}{35}$$

$$P(A \cap B \cap C) = \frac{3}{35}$$

## Probability Unit – Completing a Venn Diagram

### Practice Questions

10) 20 students went on an outdoor winter trip. The students could choose to ski, snowshoe, or dogsled.

- 10 students went skiing
- 7 students went snowshoeing
- 11 students went dogsledding
- 2 students chose to both ski and snowshoe
- 4 students went snowshoeing and dogsledding
- 3 students went skiing and dogsledding.

a) Complete a Venn diagram to represent this scenario.

b) If one student was chosen at random, what is the probability that the student did all three activities?

11) An English teacher took a class survey to determine what books the students had read. Of the 24 students in the class:

- 11 students had read *The Absolutely True Diary of a Part-Time Indian*
- 7 students had read *When Everything Feels Like the Movies*
- 10 students had read *The Hate U Give*
- 3 students had read *When Everything Feels Like the Movies* and *The Absolutely True Diary of a Part-Time Indian*
- 4 students had read *The Absolutely True Diary of a Part-Time Indian* and *The Hate U Give*
- 1 student had read *The Hate U Give* and *When Everything Feels Like the Movies*
- 1 student had read all three books

a) Complete a Venn diagram to represent this scenario.

b) If one student was chosen at random, what is the probability that the student had not read any of the books?

Probability Unit – Contingency Tables

**CONTINGENCY TABLES**

**Contingency tables** are useful when pieces of information fall in two categories, and each category has subcategories. Let’s say question 1 had two options: A and B, and question 2 had 2 options, X and Y. A contingency table allows us to present the results of both questions, as well as how the two questions relate to each other. That is, how many people chose  $A \cap X, A \cap Y, B \cap X, B \cap Y$ .

Ex: Dr. James asked her class of 25 two questions: 1) Which do you prefer: blue, red, yellow? 2) Do you like summer or winter better? The results are presented in the contingency table below

Season ↓ Colour ↓	→	BLUE	YELLOW	RED	TOTAL
SUMMER		7	2	6	15
WINTER		3	6	1	10
TOTAL		10	8	7	25

From a contingency table, we can determine probabilities.

$$P(B \cap S) = \frac{7}{25}$$

$$P(R \cap W) = \frac{1}{25}$$

$$P(S) = \frac{15}{25}$$

$$P(Y) = \frac{8}{25}$$

Practice Questions

12) A group of 115 students were surveyed about what board games they would like to play during a game night, and the snacks they would like to eat. The results are presented in the contingency table below.

	Pizza Rolls	Chips and Dip	Cookies	TOTAL
Scrabble	10	3	12	25
Trivial Pursuit	8	14	7	29
Monopoly	14	17	7	38
Risk	12	7	4	23
TOTAL	44	41	30	115

a) If a student was chosen at random, what is the probability they want to play Scrabble and want to eat Cookies?

b) If a student was chosen at random, what is the probability they want to play Risk?

c) If a student was chosen at random, what is the probability they want to eat Pizza Rolls or Chips and Dip?

## Probability Unit – Conditional Probability

### CONDITIONAL PROBABILITY

Conditional probability is the probability that an event will occur, given that another event has already occurred. We can determine the conditional probability using:

$$P(B \text{ given } A) = P(B|A) = \frac{P(A \cap B)}{P(A)} \text{ where } P(A) \neq 0$$

$$\text{Or } P(B|A) \times P(A) = P(A \cap B)$$

Ex: A 6-sided die is rolled and the top face is observed. What is the probability of obtaining a number greater than 2 given that it is an odd number?

Let Event A be obtaining an odd number

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

Let Event B be obtaining a number greater than 2

$$P(B) = \frac{4}{6} = \frac{2}{3}$$

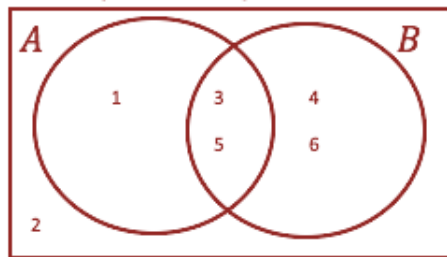
So obtaining an odd number greater than 2 is  $(A \cap B)$

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

And the probability of obtaining a number greater than 2 given that it is an odd number is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \left(\frac{1}{3}\right) \left(\frac{2}{1}\right) = \frac{2}{3}$$

$$\Omega = \{1,2,3,5,6\}$$



We can also determine the conditional probability if we have a Venn diagram. We know that we are only dealing with the cases of odd numbers, so we have 3 possible outcomes. Of those possible outcomes, 2 are greater than 2, so

$$P(B|A) = \frac{2}{3}$$

Ex: A math teacher gave her class two tests. 62% of the class passed both tests and 80% of the class passed the first test. What percent of those who passed the first test also passed the second test?

Let's re-write the question: What is the probability a student passed the second test given they passed the first test?

$$\text{Event A: passed the first test } P(A) = \frac{80}{100} = 0.8$$

$$\text{Event B: passed the second test } P(B) = ?$$

$$\text{Passed both tests: } P(A \cap B) = \frac{62}{100} = 0.62$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.62}{0.8} = 0.775$$

Therefore, the percentage of students who passed the first test who also passed the second test is 77.5%

## Probability Unit – Conditional Probability

Just like we can determine conditional probability from a Venn diagram with almost no calculations, we can also determine conditional probability if we have a contingency table.

Ex: Dr. James asked her class of 25 two questions: 1) Which do you prefer: blue, red, yellow? 2) Do you like summer or winter better? The results are presented in the contingency table below

Season ↓ Colour →	BLUE	YELLOW	RED	TOTAL
SUMMER	7	2	6	15
WINTER	3	6	1	10
TOTAL	10	8	7	25

a) If a student is chosen at random, what is the probability that the student prefers yellow given that the student likes winter best?

“Given that” means we are only looking at those students who like winter best (a total of 10 students). Of those 10 students, 6 of them prefer yellow.

$$P(Y|W) = \frac{6}{10} \text{ or } \frac{3}{5}$$

## Practice Questions

13) Canadian tourists coming back from a South American trip were interviewed about the countries they visited. Among the 40 tourists in the group:

- 20 visited Argentina
- 30 visited Brazil
- 12 visited both Argentina and Brazil

a) Given that a tourist visited Argentina, what is the probability they also visited Brazil?

b) Given that a tourist visited Brazil, what is the probability they also visited Argentina?

c) What is the probability a tourist visited Argentina given they did not visit Brazil?

d) What is the probability a tourist visited Brazil given they did not visit Argentina?

Probability Unit – Conditional Probability

14) Below is some information regarding the probability of several events.

$$P(A) = 0.31$$

$$P(B) = 0.52$$

$$P(A \cup B) = 0.67$$

$$P(C) = 0.46$$

$$P(A \cap C) = 0.13$$

$$P(B \cap C) = 0$$

Calculate:

a)  $P(A|B)$

b)  $P(B|A)$

c)  $P(A|C)$

d)  $P(C|A)$

e)  $P(C|B)$

f)  $P(B|C)$

15) A group of 115 students were surveyed about what board games they would like to play during a game night, and the snacks they would like to eat. The results are presented in the contingency table below.

	Pizza Rolls	Chips and Dip	Cookies	TOTAL
Scrabble	10	3	12	25
Trivial Pursuit	8	14	7	29
Monopoly	14	17	7	38
Risk	12	7	4	23
TOTAL	44	41	30	115

Calculate:

a)  $P(\text{Scrabble}|\text{Cookies})$

b)  $P(\text{Pizza Rolls}|\text{Risk})$

d)  $P(\text{Monopoly}|\text{Chips and Dip})$

e)  $P(\text{Cookies}|\text{Trivial Pursuit})$

## Probability Unit – Filling in a Contingency Table

We now have enough information to fill in an incomplete Contingency Table if we are given some additional information and use that to determine probabilities.

Ex: A school is hosting a trip for grade 10 and grade 11 students. The possible destinations are: Banff, Vancouver, and Halifax. The teachers collected the following information.

	Banff	Vancouver	Halifax	TOTAL
Grade 10	90	100	60	250
Grade 11	210	125	165	500
TOTAL	300	225	225	750

Additionally, the teachers know the following:

- Given that a student is in Grade 10, the probability they choose Banff is  $\frac{9}{25}$
- The probability that the student is in Grade 11 given that they chose Vancouver is  $\frac{5}{9}$

a) What is the probability that a student, chosen at random, is in grade 10 and choose Halifax?

b) Given that a student chose the Banff trip, what is the probability that they are in Grade 10?

$$\begin{array}{llll}
 \text{1) Total in Grade 11} & \text{3) Grade 10 and Banff} & \text{5) Grade 11 and Vancouver} & \text{7) Grade 10 and Halifax} \\
 = 750 - 250 = 500 & = 9/25 \times 250 = 90 & = 5/9 \times 225 = 125 & = 250 - 90 - 100 = 60
 \end{array}$$

$$\begin{array}{llll}
 \text{2) Total for Halifax} & \text{4) Grade 11 and Banff} & \text{6) Grade 10 and Vancouver} & \text{8) Grade 11 and Halifax} \\
 = 750 - 300 - 225 = 225 & = 300 - 90 = 210 & = 225 - 125 = 100 & = 500 - 210 - 125 = 165
 \end{array}$$

Therefore:

$$\text{a) } P = \frac{60}{750} = \frac{2}{25}$$

$$\text{b) } P = \frac{90}{300} = \frac{3}{10}$$

## Probability Unit – Filling in a Contingency Table

### Practice Question

16) A school is planning the schedule for the following year. Students in Grade 11 must choose two options. They must choose an art (Art, Music, or Drama) and another option (Home Economics, Biology, or Law). The students returned their options sheets and we know the following:

- 50 students chose Music
- There are a total of 200 students entering grade 11 next year
- The probability that a student chose Art is  $\frac{3}{5}$
- The probability that a student chose Home Economics is 0.3
- Given that a student chose Home Economics, the probability that the student chose Art is  $\frac{1}{4}$
- The probability that a student chose Law, given that the student chose Art, is  $\frac{2}{3}$
- The probability that a student chose Music, given that the student chose Home Economics is  $\frac{5}{12}$
- No students chose both Drama and Law
- The probability a student chose Music and Law is 0.05

a) Complete a contingency table for the above scenario.

b) What is the probability that a student chose Biology?

c) What is the probability that a student chose both Drama and Home Economics?

d) Given a student chose Biology, what is the probability the student chose Art?

e) What is the probability that a student chose Home Economics, given that the student chose Drama?



Probability Unit – Random Experiments with Several Steps

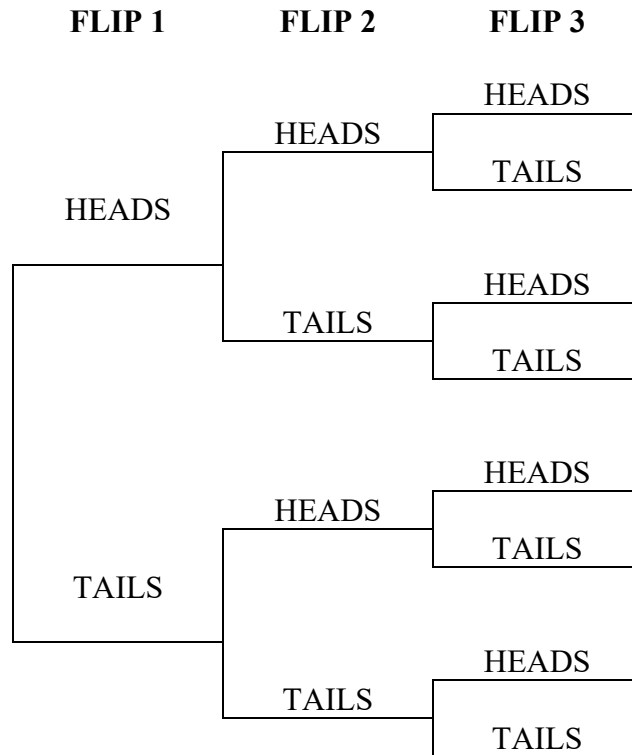
**RANDOM EXPERIMENTS WITH SEVERAL STEPS (TREE DIAGRAMS)**

Tree diagrams can also help us determine the probability of an event when that event has multiple steps. To create a tree diagram, we map out the possible options for the first step, then the second, etc. until all possible options are included. For example, let's create a tree diagram to map out the options of flipping a coin 3 times:

Step 1: List options (in this case, heads or tails)

Step 2: Following each of the results in the first step, list all possible options for the second step (in this case, heads or tails)

Step 3: Following each of the results in the second step, list all possible options for the third step (in this case, heads or tails). Continue adding steps and branches to the tree as needed.

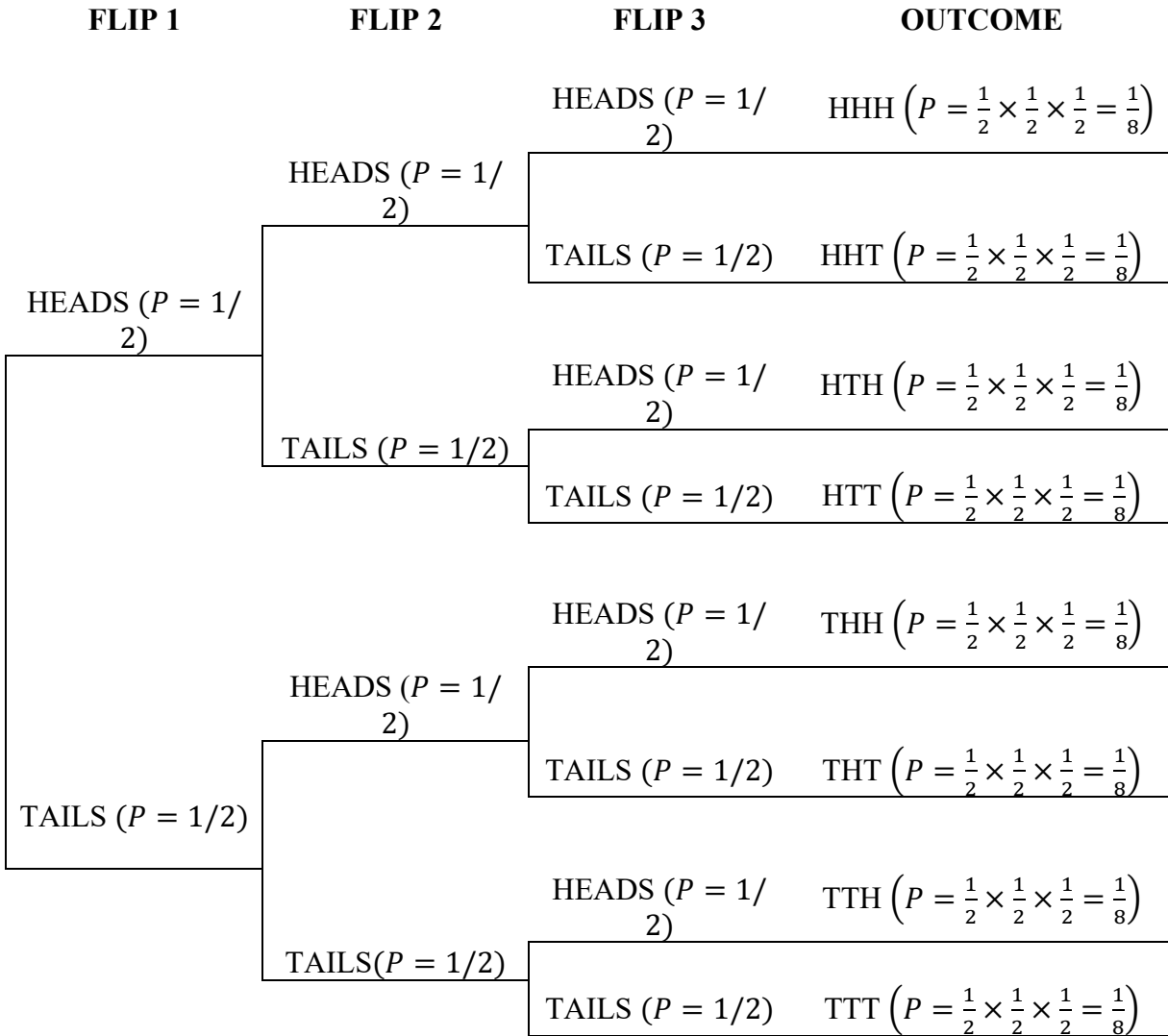


Probability Unit – Random Experiments with Several Steps

We can then add probabilities to each step in order to determine the probability of any particular outcome. The probability of any given outcome is the product of each result needed to obtain that outcome.

$$P(ABC) = P(A) \times P(B) \times P(C)$$

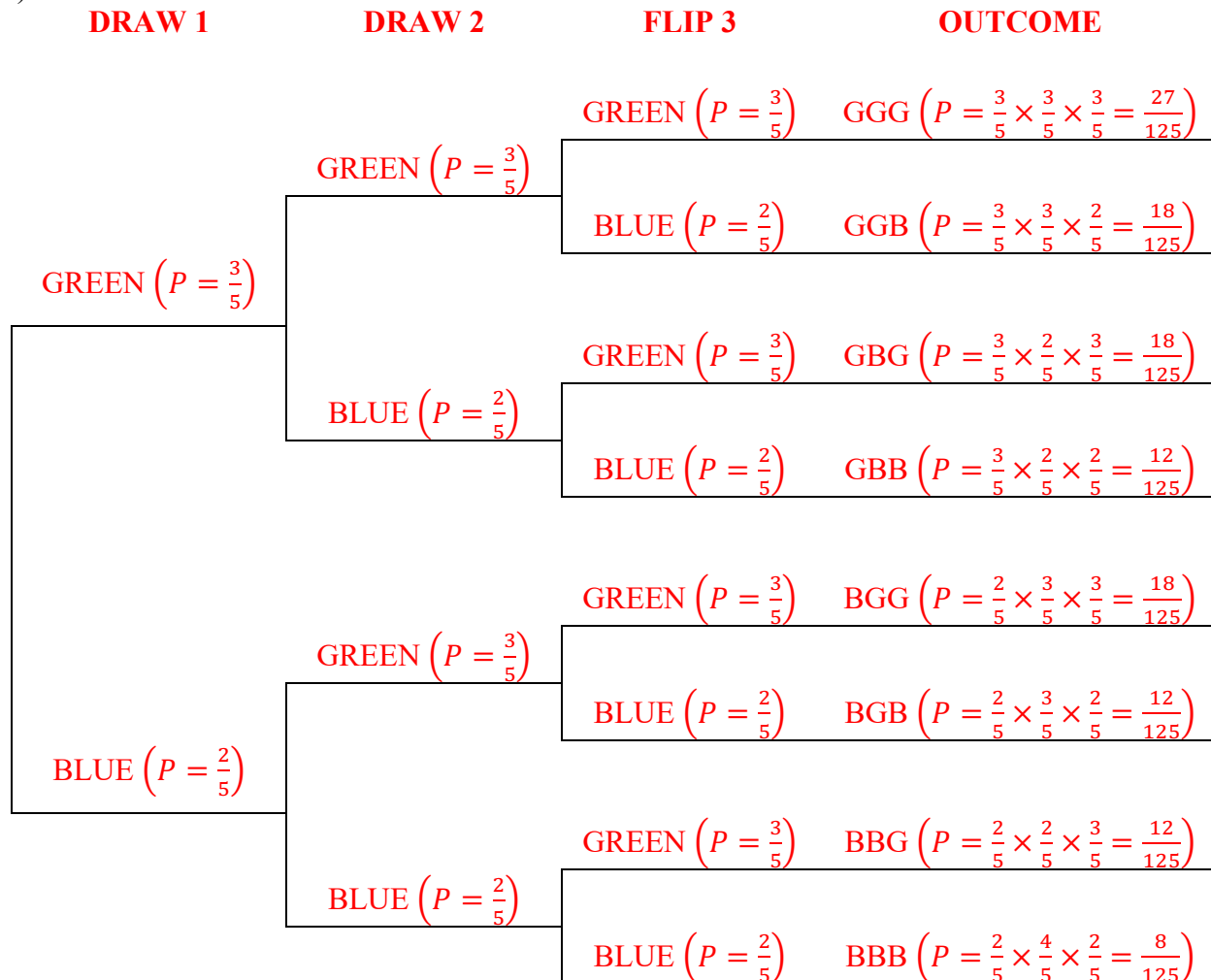
Keep in mind that in this case, the probability of heads is the same as the probability for tails, but that need not need be the case.



Probability Unit – Random Experiments with Several Steps

**Independent Events** are events in which the occurrence of one event does not influence the probability of the other happening.

Ex: You are drawing marbles from a bag that contains 3 green marbles and 2 blue marbles. After you draw each marble you put it back in the bag. If you draw three marbles, what is the probability of drawing a green marble on the first draw, followed by a blue marble, followed by a green marble (that is, what is the probability of GBG)?



Therefore,  $P(GBG) = \frac{18}{125}$

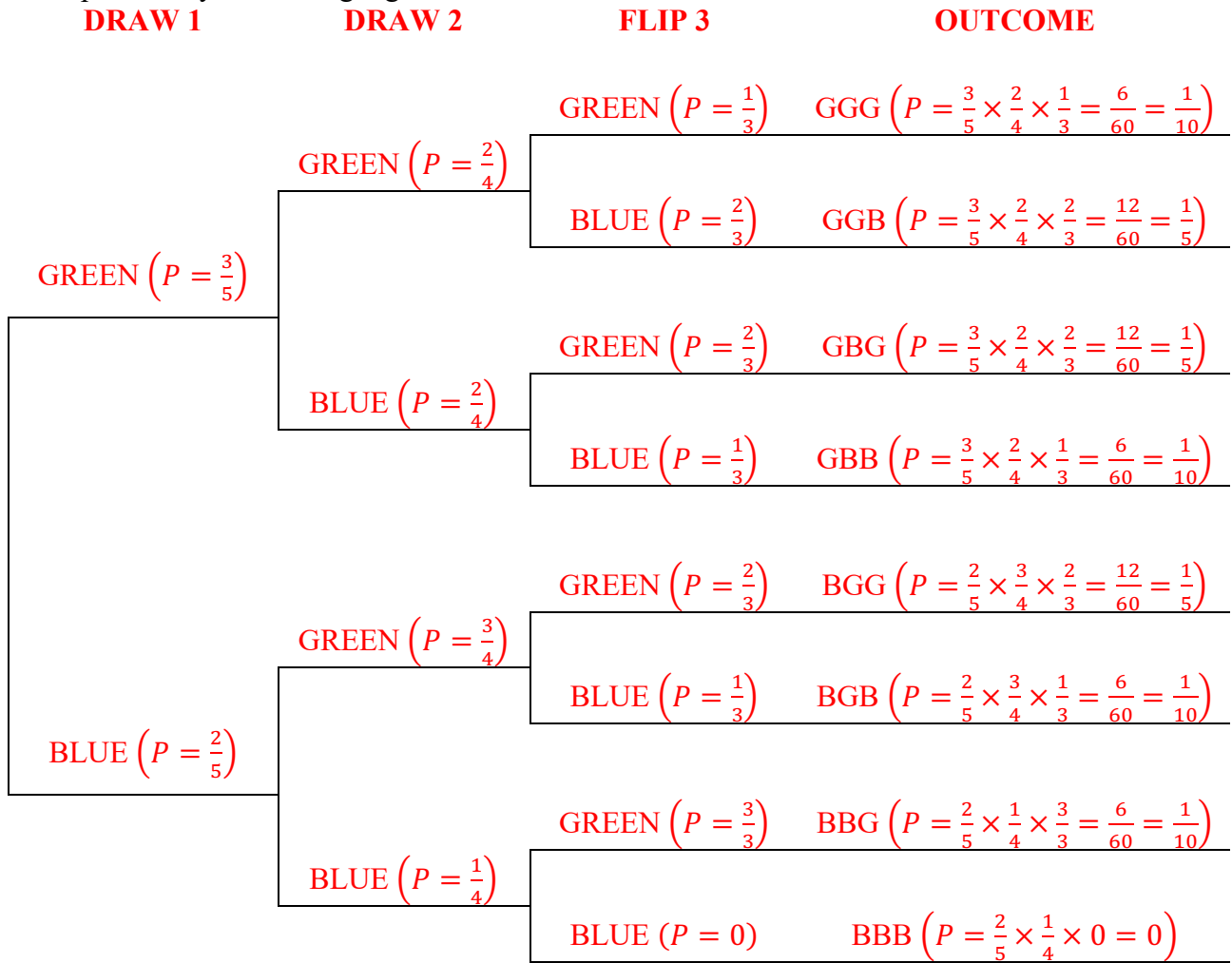
Probability Unit – Random Experiments with Several Steps

**Dependent Events** are events where the occurrence of one event does change the probability of the other happening.

Ex: You are drawing marbles from a bag that contains 3 green marbles and 2 blue marbles. After you draw each marble you do not put it back in the bag.

If you draw three marbles, what is the probability of drawing a green marble on the first draw, followed by a blue marble, followed by a green marble (that is, what is the probability of GBG)?

What is the probability of drawing 2 green marbles?



Therefore,  $P(GBG) = \frac{1}{5}$

The probability of drawing 2 green marbles is the sum of all the cases where you have 2 green marbles.

Therefore,  $P(2 \text{ green}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$

Probability Unit – Random Experiments with Several Steps

Practice Questions

17) A jar contains 5 red marbles, 3 yellow marbles, and 2 black marbles. Three marbles are drawn consecutively. Calculate the following probabilities:

a) one marble of each color is drawn if the marbles are drawn with replacement.

b) two red marbles and one black marble are drawn without replacement.

c) three marbles of the same color are drawn without replacement.

d) two black marbles and one yellow marble is drawn with replacement.

Probability Unit – Combinations and Permutations

**PERMUTATIONS**

Often, when we are talking outside of math class, we use the word combination in two different ways.

- We might say: **“My fruit salad has a combination of apples, grapes, and bananas.”** But we could just as easily say: “My fruit salad has a combination of bananas, apples, and grapes” and we would be talking about the same fruit salad. When we list the fruit, **order does not matter.**
- We might say: **“The combination to the safe is 472.”** If we said: “The combination to the safe is 247” that would be different. When we give this list of numbers, **order does matter.**

We will distinguish between these two possibilities:

- A **combination** is when **order doesn’t matter.**
- A **permutation** is when **order does matter.**

Additionally, for each of these cases, sometimes a repetition will be allowed and sometimes a repetition will not be allowed. For example, maybe the lock is designed so that once a number is used it cannot be used again.

Below we will discuss each of the 4 cases:

	<b>Order?</b>	<b>Replacement?</b>	<b>Formula</b> <i>n</i> is the number of things to choose from and we are choosing <i>r</i> times	<b>Calculator</b>
<b>Permutation</b>	It matters	Allowed	$n^r$	Use exponent button
	It matters	Not Allowed	$\frac{n!}{(n-r)!}$	nPr button
<b>Combination</b>	Doesn’t matter	Allowed	$\frac{n!}{r!(n-r)!}$	nCr button
	Doesn’t matter	Not Allowed	$\frac{(r+n-1)!}{r!(n-r)!}$	Use algebra and the ! button on your calculator

**Permutations**

A **permutation** is an **ordered** combination. There are two types of permutations

- **Replacement is allowed** – In the safe example above, the “combination” to unlock the safe could be 444 – even though the 4 was used as the first number, it can be re-used.
- **Replacement is not allowed** – If there are 3 people running a race and we are interested in the possible ways they might place. If a person came in first, that person could not also come in second or third.

## Probability Unit – Combinations and Permutations

### Permutations with Replacement

When a thing has  $n$  different types, we have  $n$  choices each time.

If there are 4 marbles in a bag, one red, one blue, one green, and one yellow, and after we draw a marble we put it back in the bag (drawing with replacement) and we are interested in all the different results we might get if we draw two times, there are 4 marbles to choose from the first time, and 4 marbles to choose from the second time. To calculate the total number of possibilities:

$$4 \times 4 \text{ or } 4^2$$

↑  
The number of

← The number of times we ...

*Ex: In the lock example above, there are 10 digits to choose from  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  and we choose 3 of them.*

$$10 \times 10 \times 10 = 10^3 = 1\,000$$

*So there are 1 000 permutations – that is, there are 1 000 different arrangements of 3 digits given that order matters and we can repeat numbers.*

In general, when repetition is allowed (with replacement) and order matters, the formula is:

$$\text{number of permutations (with replacement)} = n^r$$

where  $n$  is the number of things to choose from and we are choosing  $r$  times.

*Order matters and repetition is allowed.*

### Permutations without Replacement

In this case, there are fewer possibilities, because we cannot re-use the number once it's been used.

In general, when order matters and repetition is not allowed (without replacement), we can determine the number of possibilities using the following:

$$(\# \text{ of elements for 1st position}) \times (\# \text{ of elements for 2nd position}) \times \dots \\ \times (\text{number of elements for last position})$$

We can use something called the factorial function to write this more easily. The factorial function (symbol: !) just means to multiply a series of descending natural numbers. (The ! button is also on most calculators).

*Ex:*  $4! = 4 \times 3 \times 2 \times 1 = 24$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5\,040$$

We can write the formula for determining the number of possibilities when order matters and repetition is not allowed (without replacement) using the formula:

$$\frac{n!}{(n-r)!}$$

Where  $n$  is the number of things to choose from and we are choosing  $r$  times.

*Ex: A group of 10 people are running a race. How many possible ways could first, second, and third place be awarded?*

*We have a group of 10, order matters, and there can be no replacement. We are choosing 3 of the 10.*

*Therefore, the number of possibilities is:*

$$\frac{n!}{(n-r)!} = \frac{10!}{(10-3)!} = \frac{10!}{(7)!} = \frac{3\,628\,800}{5\,040} = 720$$

*Note: You can also use the  $nPr$  button on your calculator. Type the number of possible elements you are choosing between (in this case 10) then  $nPr$ , then the number of elements that will appear (in this case 3).*

$$10[nPr]3 = 720$$

## Probability Unit – Combinations and Permutations

### Practice Questions

18) For an electronics project, Keyla must place 6 LEDs next to one another in a line. Keyla has packages of red, green, yellow, and blue LEDs. How many different possible arrangements could Keyla make?

19) You have placed 15 different songs into your favorites list on your phone. Your phone will randomly generate playlists of 6 songs where no song is repeated within each playlist. How many different playlists can your phone generate?

### Combinations without Replacement

For combinations without repetition, think about a lottery where numbers are drawn one at a time and if we have all the numbers that are drawn (regardless of the order), we win!

We can write the formula for determining the number of possibilities when order does not matter and repetition is not allowed (without replacement) using the formula:

$$\frac{n!}{r!(n-r)!}$$

Where  $n$  is the number of things to choose from and we are choosing  $r$  times.

*Note: You can also use the  $nCr$  button on your calculator. Type the number of possible elements you are choosing between then  $nPr$ , then the number of elements that will appear.*

Ex: There are 30 students in a classroom and the teacher wants to create one group of 3 students to decorate the bulletin board. How many different possible groups are there?

In this question, there are 30 students, so 30 things to choose among. So  $n = 30$ .

The teacher is choosing 3 students, so  $r = 3$ .

$$\begin{aligned} & \frac{n!}{r!(n-r)!} \\ &= \frac{30!}{3!(30-3)!} \\ &= 4060 \end{aligned}$$

$$\text{Or } 30 [nCr] 3 = 4060$$

There are 4060 different possible groups the teacher could create.



## Probability Unit – Combinations and Permutations

### Combinations with Replacement

For combinations with replacement, think about going for ice cream. The ice cream shop offers 10 different flavors of ice cream and you are going to get 3 scoops in a dish. In this case, maybe you want all one flavor. Maybe you want 2 scoops of 1 flavor and 2 of another. Or maybe you want 3 scoops of different flavors. In this case it won't matter which scoop is dished out first, second, or third, and there is replacement, because the same flavor can be used more than once.

We can write the formula for determining the number of possibilities when order does not matter and repetition is not allowed (without replacement) using the formula:

$$\frac{(r + n - 1)!}{r!(n - r)!}$$

Where  $n$  is the number of things to choose from and we are choosing  $r$  times.

Ex: The ice cream shop offers 10 different flavors of ice cream and you are going to get 3 scoops in a dish.

How many different combinations of ice cream could you create? If 3 scoops were randomly selected, what is the probability that all 3 scoops are the same flavor?

$$\begin{aligned} & \frac{(r + n - 1)!}{r!(n - r)!} \\ &= \frac{(3 + 10 - 1)!}{3!(10 - 3)!} \\ &= \frac{479\,001\,600}{6(5040)} \\ &= \frac{479\,001\,600}{30\,240} \\ &= 15\,840 \end{aligned}$$

There are 15 840 possible combinations

To determine probability, we have to think about how many ways there could be 3 scoops of the same flavor. Since there are 10 flavors, there are 10 possibilities. Then we use what we know about probability:

$$P = \frac{10}{15840} = \frac{1}{1584}$$

Probability Unit – Combinations and Permutations

Practice Questions

20) Dr. James is putting together treats for her class. She has purchased bags of five different flavored candies: cherry, grape, orange, green apple, and pink lemonade. Dr. James randomly gives 3 candies to each student.

a) How many different combinations of candies are possible?

b) What is the probability that a student would receive 3 grape candies?

c) What is the probability that a student would receive 3 candies of the same flavor?

21) 10 marbles are placed in a bag. The marbles are numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. A student is asked to randomly pick 2 marbles simultaneously.

a) How many different combinations of numbers could the student pick?

b) What is the probability that a student would pick two consecutive numbers (ex: 1 and 2 or 6 and 7)?

Probability Unit – Odds, Expectation, and Fairness

**ODDS, EXPECTATION, AND FAIRNESS**

In addition to talking about probability, we can also talk about something called **odds**. These concepts are related. We distinguish between **odds for** and **odds against**.

The **odds for** a particular event to occur is the ratio of the number of favorable outcomes to the number of unfavorable outcomes.

<b>Odds for Event A</b>
$\text{Odds for Event A} = \frac{\text{The number of ways event A occurs}}{\text{The number of ways A does not occur}}$
OR
$= \frac{\text{Number of favorable outcomes}}{\text{Number of unfavorable outcomes}}$
OR
<i>Number of favorable outcomes: Number of unfavorable outcomes</i>

*Ex: The odds that a football team will win the championship are 3:2. This means that the team has 3 chances of winning and 2 chances of losing.*

The **odds against** a particular event to occur is the ratio of the number of unfavorable outcomes to the number of favorable outcomes.

<b>Odds against Event A</b>
$\text{Odds against Event A} = \frac{\text{The number of ways event A does not occur}}{\text{The number of ways A occurs}}$
OR
$= \frac{\text{Number of unfavorable outcomes}}{\text{Number of favorable outcomes}}$
OR
<i>Number of unfavorable outcomes: Number of favorable outcomes</i>

*Ex: The odds against a player who bets on “0” in roulette are 36:1. That means the player has 36 chances of losing and 1 chance of winning.*

**Difference between probability and odds**

Probability of Event A: P(A)
$P(A) = \frac{\text{The number of ways event A can occur}}{\text{The total number of possible outcomes}}$
$= \frac{\text{number of favorable outcomes}}{\text{number of favorable outcomes} + \text{number of unfavorable outcome}}$

*Ex: The odds that the basketball team will win the next game are 4 to 3. What is the probability that the basketball team will NOT win the next game?*

Odds for are 4:3 (4 chances to win, 3 chances to lose)

So the odds against are 3:4 (3 chances to lose, 4 chances to win)

$$\text{Probability of not winning} = \frac{\text{number associated with not winning}}{\text{total numer}} = \frac{3}{3 + 4} = \frac{3}{7}$$

Therefore, the probability of not winning is  $\frac{3}{7}$ .

Practice Questions

22) The probability that the local basketball team will win the next game is estimated at  $\frac{2}{7}$ .

- a) What are the odds for the team winning the next game?
- b) What are the odds against the team winning the next game?

23) Two dice (numbered 1 to 6) are rolled. What are:

- a) the odds for obtaining a sum of 7?
- b) the odds against obtaining a sum of 12?
- c) the odds for obtaining a sum of 2?
- d) the odds for obtaining a sum of 1?

## Probability Unit – Odds, Expectation, and Fairness

### Mathematical Expectation

We can use mathematical **expectation** to determine how much money a person would expect to win (or lose) when placing a bet, given they know the probability of winning (and of losing). We also use the term “**expected gain**” interchangeably with expectation.

- If the result is positive, the person can expect to make a profit.
- If the result is negative, the person can expect to lose money.

$$\text{Expected gain} = (\text{probability of winning}) \times (\text{net gain}) + (\text{probability of losing}) \times (\text{loss})$$

$$\begin{array}{ccc} & \uparrow & \uparrow \\ & \text{Gain - initial bet} & \text{initial bet (written as negative)} \end{array}$$

*Ex: In a game of roulette, a person who has bet on the winning number receives 35 times the amount of the bet. Since the roulette wheel slots are numbered 0 to 36, the probability that the ball will land on a given number is  $\frac{1}{37}$ .*

*Determine the expected gain for a bet of \$10.*

$$\text{Probability of winning} = \frac{1}{37}$$

$$\text{Probability of losing} = \frac{36}{37}$$

$$\text{Initial Bet (loss)} = -\$10$$

$$\text{Winnings} = 35 \times 10 = 350$$

$$\text{Net gain} = 350 - 10 = 340$$

$$\text{Expected gain} = (\text{probability of winning}) \times (\text{net gain}) + (\text{probability of losing}) \times (\text{loss})$$

$$\text{Expected gain} = \left(\frac{1}{37}\right) \times (340) + \left(\frac{36}{37}\right) \times (-10)$$

$$\text{Expected gain} = -\$0.54$$

Thus, in general, a person can expect to lose \$0.54 playing roulette. We know that any given time a person plays, they will either lose \$10 (in this example) or win \$350 (and once the initial bet is considered, they would come away winning \$340), but if played an infinite number of times, a person could expect to lose \$0.54.

Probability Unit – Odds, Expectation, and Fairness

Practice Questions

24) A carnival game costs \$2 to play. In this game, there is a jar containing 2 black marbles and 3 white marbles. If you draw a black marble, you get \$4. If you draw a white marble you get nothing. What is the expected gain from this game?

25) Consider the following game. Toss a coin twice. You lose your initial bet of \$10 if you do not get tails on either toss. You win \$4 and get your initial bet back if you get tails on one toss. You win \$8 and get your initial bet back if you get tails on both tosses. What is the expected gain from this game?

## Probability Unit – Odds, Expectation, and Fairness

### Fairness

- When the mathematical expectation (or expected gain) of a game is greater than 0, it is advantageous to the person playing.
- When the mathematical expectation (or expected gain) of a game is less than 0, it is disadvantageous to the person playing.
- When a game's mathematical expectation is equal to 0, the game is considered **fair**. That is, given a person plays an infinite number of times, they can expect to lose nothing and gain nothing.

*Ex: A game consists of drawing a marble at random from a set composed of 5 red marbles and 4 blue marbles. It costs a person \$10 to place a bet. If a blue marble is drawn, the person wins \$12.50 plus their initial bet. If a red marble is drawn, the \$10 bet is lost. Is this game fair?*

$$\text{Probability of winning} = \frac{4}{9}$$

$$\text{Probability of losing} = \frac{5}{9}$$

$$\text{Initial Bet (loss)} = -\$10$$

$$\text{Winnings} = \$12.50 + \$10 = \$22.50$$

$$\text{Net gain} = \$22.50 - 10 = \$12.50$$

$$\text{Expected gain} = (\text{probability of winning}) \times (\text{net gain}) + (\text{probability of losing}) \times (\text{loss})$$

$$\text{Expected gain} = \left(\frac{4}{9}\right) \times (12.50) + \left(\frac{5}{9}\right) \times (-10)$$

$$\text{Expected gain} = \$0.00$$

**Therefore, this game is fair.**

*Ex: For \$2, a person can participate in a draw for a \$50 gift card. The probability of winning the draw is  $\frac{1}{100}$ . Is this draw fair?*

$$\text{Probability of winning} = \frac{1}{100}$$

$$\text{Probability of losing} = \frac{99}{100}$$

$$\text{Initial Bet (loss)} = -\$2$$

$$\text{Winnings} = \$50.00$$

$$\text{Net gain} = \$50.00 - \$2.00 = \$48.00$$

$$\text{Expected gain} = (\text{probability of winning}) \times (\text{net gain}) + (\text{probability of losing}) \times (\text{loss})$$

$$\text{Expected gain} = \left(\frac{1}{100}\right) \times (48) + \left(\frac{99}{100}\right) \times (-2)$$

$$\text{Expected gain} = -\$1.50$$

**Therefore, this game is not fair.**

Probability Unit – Odds, Expectation, and Fairness

Practice Questions

26) In a game at a carnival, a person must draw a card from a deck of 54 cards (the deck contains both jokers). If the person draws one of the four aces, they win a box of candy valued at \$10. If it costs \$2 to play, is this game fair?

27) In a game at a carnival, a person must draw a card from a deck of 54 cards. If the person draws one of the four aces, they win a box of candy valued at \$10. If the person draws one two jokers, they win a teddy bear valued at \$40. If the person draws any other card, they do not win anything.

a) If it costs \$2 to play, is this game fair?

b) If the game is not fair, how much should the game cost to play in order for it to be fair?



## Probability Unit – Weighted Mean

### WEIGHTED MEAN

Remember that the mean of a set of numbers is the average. To determine mean:

$$\text{Mean} = \frac{\text{sum of data values}}{\text{total number of values}}$$

Add all the numbers together and divide by how many numbers there are.

Ex: A student received the following grades on their math quizzes: 67%, 72%, 85%, 49% and 65%. What is the student's average quiz grade in math?

$$\begin{aligned}\text{Mean} &= \frac{\text{sum of data values}}{\text{total number of values}} \\ \text{Mean} &= \frac{67 + 72 + 85 + 49 + 65}{5} \\ \text{Mean} &= 67.6\%\end{aligned}$$

For a weighted mean, some values are more important than others.

To calculate weighted mean:

- Convert the weight of each item from a percent to a decimal (divide by 100)
- Multiply the value of each item by the weight
- Add (values x weight) together

Ex: A student has the following marks in math.

Item	Grade (%)	Weight (%)
Test 1	72%	50
Quiz 1	64%	25
Homework 1	81%	25

What is the student's overall math mark?

1. Convert weights into decimals

$$\frac{50}{100} = 0.5$$

$$\frac{25}{100} = 0.25$$

$$\frac{25}{100} = 0.25$$

2. Multiply values by weights

$$72 \times 0.5 = 36$$

$$64 \times 0.25 = 16$$

$$81 \times 0.25 = 20.25$$

3. Add all values together

$$\text{Mean} = 36 + 16 + 20.25$$

$$\text{Mean} = 72.25$$

Therefore, the student's average math mark is 72.25%

Probability Unit – Weighted Mean

Practice Questions

28) In order to calculate final grades in a class, the teacher combines assignment grades, test grades, and quiz grades. Assignments are worth 30% of the final grade, tests are worth 50%, and quizzes are worth 20%. If a student has an assignment grade of 82%, a test grade of 65%, and a quiz grade of 74%, what is the student's final grade?

29) Nico wants to buy a new car and has decided to rate each car according to the following criteria and values:

- Appearance: 15%
- Reliability: 40%
- Efficiency: 15%
- Comfort: 30%

The following is a list of possible cars and the ratings of each (0 is the lowest mark and 10 is the highest).

	Appearance	Reliability	Efficiency	Comfort
Car A	5	9	8	6
Car B	6	10	8	4
Car C	4	7	3	9

Nico has decided to buy the car with the highest average rating. Which car should Nico buy?

## Voting Procedures – Plurality

There are a lot of different ways to count votes in order to determine a winner.

- Canada, the United States, and many countries in Africa use **plurality**.
- The Australian House of Representative, the Indian presidential election, and parliament in Papua New Guinea use **majority with elimination**.
- Nauru, a tiny island country in Micronesia (just northeast of Australia) uses **Borda count**.
- The **Condorcet method** is not practical to use in elections, but a modified version of it is the basis for voting in Robert's Rules of Order.
- Most countries in western Europe and South America, as well as Russia, Kazakhstan, and several countries in Africa use **proportional representation**.

Each method has advantages and disadvantages.

### Plurality

In voting under a plurality method, people vote for their favorite option. The option with the most votes is the winner.

Advantages: it is easy to calculate and a winner is guaranteed (unless there is a tie)

Disadvantages: A winner could emerge that most people do not want

Ex: Grade 11 students are electing their class president. Of the 150 students in grade 11, 70 vote for Ali, 30 vote for Brenda, and 50 vote for Chris. Under plurality, who wins the election?

**Ali wins.**

In some questions, we will be given a ranked ballot where people put all the options in order.

Ex: Dr. James is going to bring a snack (apples, granola bars, or pepperoni sticks) in for her math students and the students will vote to determine the snack. The following table summarizes the results of the student vote.

# of votes \ Preference	45	32	28	23
1 <sup>st</sup> choice	David	Eli	Francis	Eli
2 <sup>nd</sup> choice	Eli	David	David	Francis
3 <sup>rd</sup> choice	Francis	Francis	Eli	David

Given these results, who would win under plurality?

**We only need to pay attention to the 1<sup>st</sup> choice votes.**

**David: 45**

**Eli: 32+23=55**

**Francis: 28**

**Therefore, Eli would win**

## Voting Procedures – Plurality

### Practice Questions

1) In a hockey league, a committee of 53 members must select a recipient for the trophy for the hardest-working player from a list of three candidates.

- Candidate A received 16 votes
- Candidate B received 20 votes
- Candidate C received 17 votes

Under plurality voting, which candidate wins?

2) To determine Monday's menu at a high school cafeteria, the school's 250 grade 11 students were asked to rank three menus in order of preference. The results are presented in the table below.

# of votes	63	51	46	45	24	21
Preference						
1 <sup>st</sup> choice	Pizza	Hamburgers	Tacos	Tacos	Hamburgers	Pizza
2 <sup>nd</sup> choice	Hamburgers	Pizza	Pizza	Hamburgers	Tacos	Tacos
3 <sup>rd</sup> choice	Tacos	Tacos	Hamburgers	Pizza	Pizza	Hamburgers

Under plurality voting, which menu item would be served?

3) A new school is being built and an election was held to determine if it will be in Village A, Village B, or Village C. Village A received 45% of the vote. Village B received 35% of the vote. Village C received 20% of the vote. Under plurality rule, where will the school be built?

## Voting Procedures – Majority

### Majority

In voting under a majority method, people vote for their favorite option. The option with more than half the votes is the winner.

Step 1: determine the total number of votes

Step 2: divide the total number of votes by 2

Step 3: determine the number of first place votes for each option

Step 4: determine a winner (the option with **more than half** the votes).

*\* Note: if no option has more than half the votes, there is no winner.*

Advantages: more than half the people are guaranteed to like the winner

Disadvantages: there may be no winner

Ex: A group of friends is trying to decide where to eat dinner. They decide to vote for their favorite type of food, and the results are presented in the table below. Which option wins using majority?

# of votes	4	4	7	5
Preference				
1 <sup>st</sup> choice	Vietnamese	Italian	Vietnamese	Sushi
2 <sup>nd</sup> choice	Sushi	Vietnamese	Italian	Italian
3 <sup>rd</sup> choice	Italian	Sushi	Sushi	Vietnamese

Total number of votes:  $4 + 4 + 7 + 5 = 20$

Majority:  $\frac{20}{2} = 10$  So need more than 10 votes

Vietnamese:  $4 + 7 = 11$

Sushi: 5

Italian: 4

Therefore, Vietnamese wins.

Ex: Grade 11 students are electing their class president. Of the 140 students in grade 11, 70 vote for Ali, 30 vote for Brenda, and 50 vote for Chris. Under majority, who wins the election?

$$\frac{140}{2} = 70$$

No student received more than 70 votes, so there is no winner.

## Voting Procedures – Majority

### Practice Questions

4) The 40 members of the board of directors of a company must vote to elect the president of the board. Three candidates apply for the position and the results of the vote are presented in the table below.

# of votes	16	14	10
Preference			
1 <sup>st</sup> choice	A	C	B
2 <sup>nd</sup> choice	B	A	A
3 <sup>rd</sup> choice	C	B	C

Given the board of directors use majority to determine the winner, which candidate wins?

5) To determine Monday's menu at a high school cafeteria, the school's 250 grade 11 students were asked to rank three menus in order of preference. The results are presented in the table below.

# of votes	63	51	46	45	24	21
Preference						
1 <sup>st</sup> choice	Pizza	Pizza	Tacos	Tacos	Tacos	Pizza
2 <sup>nd</sup> choice	Hamburgers	Hamburgers	Pizza	Hamburgers	Hamburgers	Tacos
3 <sup>rd</sup> choice	Tacos	Tacos	Hamburgers	Pizza	Pizza	Hamburgers

Under majority voting, which menu item would be served?

Voting Procedures – Majority with Elimination

**Majority with Elimination**

In voting under a **majority with elimination** method, people rank the options from favorite to least favorite. The option with more than half the votes is the winner. However, if no option receives more than half the votes, the option with the fewest votes is eliminated and re-assigned.

Step 1: determine the total number of votes

Step 2: divide the total number of votes by 2

Step 3: determine the number of first place votes for each option

Step 4: determine a winner (the option with **more than half** the votes)

Step 5: If no winner, the option with the fewest number of votes is eliminated and the votes are re-assigned to the next option on the list. Repeat this step until an option has more than half the votes

Advantages: A winner is guaranteed and people’s preferences for all options are included

Disadvantages: Voting and counting the votes can be time-consuming depending on the number of options

Ex: A political party is holding elections to determine its new leader. There are 5 people nominated for the position.

# of votes	23	56	60	20	41
Preference					
1 <sup>st</sup> choice	A	B	C	A	D
2 <sup>nd</sup> choice	B	A	B	E	B
3 <sup>rd</sup> choice	C	C	A	C	A
4 <sup>th</sup> choice	D	D	E	D	C
5 <sup>th</sup> choice	E	E	D	B	E

Total Votes:  $23 + 56 + 60 + 20 + 41 = 200$

$\frac{200}{2} = 100$  An option needs more than 100 votes to win

1<sup>st</sup> round

A:  $23 + 20 = 43$

B: 56

C: 60

D: 41

~~E: 0~~

No majority

Eliminate E (no votes to reassign)

2<sup>nd</sup> round

A:  $23 + 20 = 43$

B: 56

C: 60

~~D: 41~~

No majority

Eliminate D. The second choice for people who voted for D is B, so B gets the votes.

3<sup>rd</sup> round

~~A:  $23 + 20 = 43$~~

B:  $56 + 41 = 97$

C: 60

No majority

Eliminate A.

23 votes reassigned to C  
20 votes reassigned to E (but E is eliminated) then to C.

4<sup>th</sup> round

B:  $56 + 41 = 97$

C:  $60 + 23 + 20 = 103$

C has a majority

C is the winner

Voting Procedures – Majority with Elimination

Practice Questions

6) The table below presents the results of an election between 3 candidates: A, B, and C.

# of votes	50	30	27	24	10
Preference					
1 <sup>st</sup> choice	C	B	A	A	B
2 <sup>nd</sup> choice	B	C	B	C	A
3 <sup>rd</sup> choice	A	A	C	B	C

Which candidate wins using majority with elimination?

7) Philemon Wright is selling grad hoodies, but can only order 1 color of hoodies. Students are asked to vote among the following options: Green, Blue, Red, and Black. The results of the vote are presented in the table below:

# of votes	52	41	35	33	29	21
Preference						
1 <sup>st</sup> choice	Blue	Green	Green	Black	Blue	Red
2 <sup>nd</sup> choice	Green	Blue	Black	Blue	Green	Green
3 <sup>rd</sup> choice	Black	Red	Red	Red	Red	Black
4 <sup>th</sup> choice	Red	Black	Blue	Green	Black	Blue

Using majority with elimination, which color hoodie will be ordered?



## Voting Procedures – Borda Count

### Borda Count

In voting under a **Borda count** method, people rank the options from favorite to least favorite. Points are assigned to each choice. For each option, the number of votes is multiplied by the points, and all are added together. The option with the most points is the winner.

Step 1: Assign points to each option (last choice is 0, next is 1, next is 2, and so on until all choices are assigned)

Step 2: Every time an option is ranked, the number of votes is multiplied by the points assigned

Step 3: All the points for each option are added together

Step 4: The option with the most votes is the winner.

Advantages: This allows for a nuanced interpretation of preferences and will generally lead to a high degree of satisfaction among the electorate

Disadvantages: It's a complicated system to implement

Ex: Dr. James is going to bring a snack (apples, granola bars, or pepperoni sticks) in for her math students and the students will vote to determine the snack. The following table summarizes the results of the student vote.

# of votes \ Preference	45	32	28	23	
1 <sup>st</sup> choice	David	Eli	Francis	Eli	× 2
2 <sup>nd</sup> choice	Eli	David	David	Francis	× 1
3 <sup>rd</sup> choice	Francis	Francis	Eli	David	× 0

Given these results, who would win under Borda count?

David:  $45(2) + 32(1) + 28(1) + 23(0) = 150$  points

Eli:  $45(1) + 32(2) + 28(0) + 23(2) = 155$  points

Francis:  $45(0) + 32(0) + 28(2) + 23(1) = 79$  points

David is the winner

Voting Procedures – Borda Count

Practice Questions

8) A group of friends is trying to decide where to eat dinner. They decide to vote for their favorite type of food, and the results are presented in the table below. Which option wins using Borda count?

# of votes	4	4	7	5
Preference				
1 <sup>st</sup> choice	Vietnamese	Italian	Vietnamese	Sushi
2 <sup>nd</sup> choice	Sushi	Vietnamese	Italian	Italian
3 <sup>rd</sup> choice	Italian	Sushi	Sushi	Vietnamese

9) To determine Monday’s menu at a high school cafeteria, the school’s 250 grade 11 students were asked to rank three menus in order of preference. The results are presented in the table below. Which option wins using Borda count?

# of votes	63	51	46	45	24	21
Preference						
1 <sup>st</sup> choice	Pizza	Pizza	Tacos	Tacos	Tacos	Pizza
2 <sup>nd</sup> choice	Hamburgers	Hamburgers	Pizza	Hamburgers	Hamburgers	Tacos
3 <sup>rd</sup> choice	Tacos	Tacos	Hamburgers	Pizza	Pizza	Hamburgers

Voting Procedures – Condorcet method

**Condorcet Method**

The Condorcet method is a voting system in which the winner is the option that, when compared in a head-to-head competition with every other method, is the preferred option.

- Step 1: Compare option A to option B. Determine the number of people who prefer A to B and the number of people who prefer B to A. The option with the most votes is the winner.
- Step 2: Compare option A to another option and repeat step 1. Do this for as many options as exist.
- Step 3: Compare option B to every other option, one option at a time (as in step 1).
- Step 4: Repeat with every possible combination of options.
- Step 5: Declare the winner – the option that never loses.

Advantages: this method is the most rigorous and assures the most number of people will be satisfied by the outcome as possible.

Disadvantages: there is often no winner and it is complicated to implement.

Ex: The 40 members of the board of directors of a company must vote to elect the president of the board. Three candidates apply for the position and the results of the vote are presented in the table below.

# of votes	16	14	10
Preference			
1 <sup>st</sup> choice	A	C	B
2 <sup>nd</sup> choice	B	A	A
3 <sup>rd</sup> choice	C	B	C

Given the board of directors use the Condorcet method to determine the winner, which candidate wins?

Compare A to B

A:  $16 + 14 = 30$   
 B: 10

A wins

Compare A to C

A:  $16 + 10 = 26$   
 B: 14

A wins

Compare B to C

B:  $16 + 10 = 26$   
 C: 14

B wins

Every time A is compared to another, A wins.  
 Therefore, A is the overall winner.

### Voting Procedures – Condorcet method

Ex: Voters are asked to choose between 4 candidates: A, B, C, and D. The results of the vote are presented in the table below:

# of votes	70	60	50	45
Preference				
1 <sup>st</sup> choice	A	D	C	B
2 <sup>nd</sup> choice	B	A	D	C
3 <sup>rd</sup> choice	C	B	A	D
4 <sup>th</sup> choice	D	C	B	A

Using the Condorcet method, which candidate wins?

Compare A and B

$$A: 70 + 60 + 50 = 180$$

$$B: 45$$

A wins/B loses

Compare B and C

$$B: 70 + 60 + 45 = 175$$

$$C: 50$$

B wins/C loses

Compare A and C

$$A: 70 + 60 = 130$$

$$C: 50 + 45 = 95$$

A wins/C loses

Compare C and D

$$C: 70 + 50 + 45 = 130$$

$$D: 60$$

C wins/D loses

Compare A and D

$$A: 70$$

$$D: 60 + 50 + 45 = 155$$

D wins/A loses

Since A, B, C, and D all lose at least once, there is no winner.

Voting Procedures – Condorcet method

Practice Questions

10) Philemon Wright is selling grad hoodies, but can only order 1 color of hoodies. Students are asked to vote among the following options: Green, Blue, Red, and Black. The results of the vote are presented in the table below:

# of votes	52	41	35	33	29	21
Preference						
1 <sup>st</sup> choice	Blue	Green	Green	Black	Blue	Red
2 <sup>nd</sup> choice	Green	Blue	Black	Blue	Green	Green
3 <sup>rd</sup> choice	Black	Red	Red	Red	Red	Black
4 <sup>th</sup> choice	Red	Black	Blue	Green	Black	Blue

Using the Condorcet method, which color hoodie will be ordered?

11) A group of friends is trying to decide where to eat dinner. They decide to vote for their favorite type of food, and the results are presented in the table below. Which option wins using the Condorcet method?

# of votes	4	4	7	5
Preference				
1 <sup>st</sup> choice	Vietnamese	Italian	Vietnamese	Sushi
2 <sup>nd</sup> choice	Sushi	Vietnamese	Italian	Italian
3 <sup>rd</sup> choice	Italian	Sushi	Sushi	Vietnamese

Voting Procedures – Proportional Representation

**Proportional Representation**

In **proportional representation**, individuals vote for the party they most prefer. The total number of votes for each party are calculated, and then the parties are awarded a number of seats in proportion to the number of votes received.

Step 1: Determine the number of votes for each party

Step 2: Determine the total number of votes

Step 3: Determine the proportion of votes for each party  $\frac{\text{votes for the party}}{\text{total number of votes}}$

Step 4: Multiply the proportion of votes for each party by the number of seats available

Step 5: Ignore the decimals (do not round) and assign each party that number of seats

Step 6: Assign any remaining seats by choosing the highest decimal, the next highest, etc until all seats are assigned

Advantages: This system is quite representative of individual preferences and leads to a distribution of power that fairly accurately reflects the will of the electorate

Disadvantages: There is often no majority party and thus requires a coalition government that can slow the decision making process.

Ex: A country is electing representatives to fill 90 seats in its Parliament. There are 5 parties in the election: A, B, C, D, E. The results of the election are presented in the table below.

Party	A	B	C	D	E
# of votes	113	108	132	86	92

Using proportional representation, how many seats will each party earn?

Total Votes:  $113 + 108 + 132 + 86 + 92 = 531$

Party	Proportion of votes	Seats available (90)	Seats for the party	Seats	Additional Seats	Final Seats
A	$\frac{113}{531} = 0.21281$	90	$0.21281 \times 90 = 19.1529$	19		19
B	$\frac{108}{531} = 0.20339$	90	$0.20339 \times 90 = 18.3051$	18		18
C	$\frac{132}{531} = 0.24859$	90	$0.24859 \times 90 = 22.3731$	22		22
D	$\frac{86}{531} = 0.16196$	90	$0.16196 \times 90 = 14.5764$	14	+1	15
E	$\frac{92}{531} = 0.17326$	90	$0.17326 \times 90 = 15.5934$	15	+1	16

Total: 88  
So 2 more needed

Voting Procedures – Proportional Representation

Ex: A town is holding an election to fill 12 seats on its council. There are 3 parties and the results of the election are in the table below:

Party	Percent of Vote Received
A	38%
B	33.6%
C	28.4%

Using proportional representation, how will the available seats be divided?

Party	Proportion of votes	Seats available (12)	Seats for the party	Seats	Additional Seats	Final Seats
A	$\frac{38}{100} = 0.38$	12	$0.38 \times 12 = 4.56$	4	+1	5
B	$\frac{33.6}{100} = 0.336$	12	$0.336 \times 12 = 4.032$	4		4
C	$\frac{28.4}{100} = 0.284$	12	$0.284 \times 12 = 3.408$	3		3

Total: 11  
So 1 more needed

## Voting Procedures – Proportional Representation

### Practice Questions

12) In 2019, Canada held a federal election and the number of popular votes each party received is in the table below.

Party	# of Votes
Liberal	6 018 728
Conservative	6 239 227
Bloc Quebecois	1 387 030
New Democratic Party	2 903 722
Green Party	1 189 607
Independent	72 546

a) If Canada used a proportional representation system, how many of the 338 seats in Parliament would each party receive?

b) The actual number of seats won by each party under our current system (districts using plurality) is in the table below. How would proportional representation change these results?

Party	# of Seats
Liberal	157
Conservative	121
Bloc Quebecois	32
New Democratic Party	24
Green Party	3
Independent	1



Voting Procedures – Approval voting

The final method of voting we will consider is Approval voting.

In approval voting, instead of ranking candidates, voters will select as many options as they like. Every option they choose will receive 1 vote. The option with the most votes is the winner.

Ex: The results of an election are presented in the table below.

Number of Voters	45	32	28	23
	A	B	A	A
	D	C	B	
		D	C	

Using approval voting, which option would win?

A:  $45 + 28 + 23 = 96$

B:  $32 + 28 = 60$

C:  $32 + 28 = 60$

D:  $45 + 32 = 77$

Therefore, A wins

Practice Questions

13) A group of friends is trying to decide where to eat dinner. They decide to vote for their favorite type of food, and the results are presented in the table below. Which option wins using Approval voting?

# of votes	4	4	7	5
	Vietnamese Sushi	Italian Vietnamese	Vietnamese Italian Sushi	Sushi

14) To determine Monday’s menu at a high school cafeteria, the school’s 250 grade 11 students were asked to rank three menus in order of preference. The results are presented in the table below.

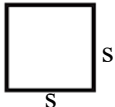
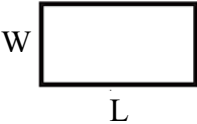
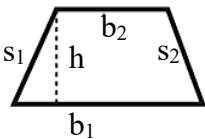
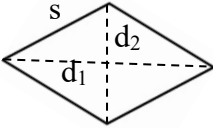
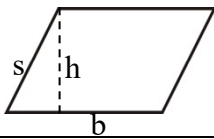
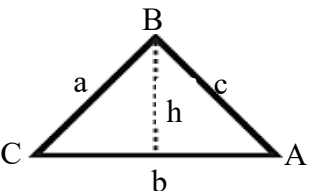
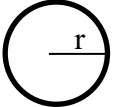
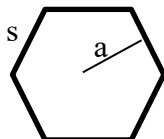
# of votes	63	51	46	45	24	21
	Pizza Hamburgers	Pizza Tacos	Tacos	Tacos Hamburgers	Hamburgers	Pizza

## Geometry – 2D Shapes: Area and Perimeter

When looking at 2-dimensional shapes, we can talk about perimeter and area.

**Perimeter** is the distance around the outside of the shape, or the distance to trace the outline of the shape.

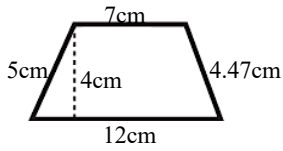
**Area** is the measure of the inside of the shape – if you were going to color in the shape, the amount of space you would have to color.

Shape	Area	Perimeter
<b>Square</b> 	$A = s^2$	$P = 4s$
<b>Rectangle</b> 	$A = L \times W$	$P = 2W + 2L$
<b>Trapezoid</b> 	$A = \frac{(b_1 + b_2) \times h}{2}$	$P = b_1 + b_2 + s_1 + s_2$
<b>Rhombus</b> 	$A = \frac{d_1 \times d_2}{2}$	$P = 4s$
<b>Parallelogram</b> 	$A = b \times h$	$P = 2b + 2s$
<b>Triangle</b> 	$A = \frac{b \times h}{2}$ $A = \frac{a \times b \times \sin C}{2}$ $A = \sqrt{s(s - a)(s - b)(s - c)}$ Where $s$ is the half the perimeter	$P = a + b + c$
<b>Circle</b>  $d = 2r$	$A = \pi r^2$	$P = 2\pi r$ or $P = \pi d$
<b>Regular “-agon”</b> (all sides are same length) (ex: pentagon, hexagon, heptagon, octagon, nonagon, decagon, etc.)  Where $a$ is the apothem	$A = \frac{P \times a}{2}$	$P = s \times (\# \text{ of sides})$

Geometry – 2D Shapes: Area and Perimeter

Ex: Find the area the perimeter of the following shapes.

a)



$$A = \frac{(12 + 7) \times 4}{2}$$

$$A = \frac{(19) \times 4}{2}$$

$$A = \frac{76}{2}$$

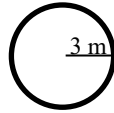
$$A = 38 \text{ cm}^2$$

$$P = b_1 + b_2 + s_1 + s_2$$

$$P = 12 + 7 + 5 + 4.47$$

$$P = 28.47 \text{ cm}$$

b)



$$A = \pi r^2$$

$$A = \pi \times 3^2$$

$$A = \pi \times 9$$

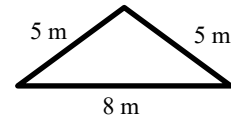
$$A = 28.27 \text{ m}^2$$

$$P = 2\pi r$$

$$P = 2\pi \times 3$$

$$P = 18.85 \text{ m}$$

c)



$$P = a + b + c$$

$$P = 5 + 5 + 8$$

$$P = 18$$

$$\text{So } s = \frac{18}{2} = 9$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$A = \sqrt{9(9-5)(9-5)(9-8)}$$

$$A = \sqrt{9(4)(4)(1)}$$

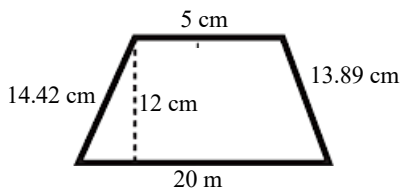
$$A = \sqrt{144}$$

$$A = 12 \text{ m}^2$$

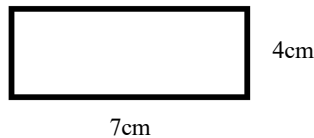
Practice questions

1. Find the area and perimeter of the following shapes

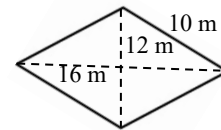
a)



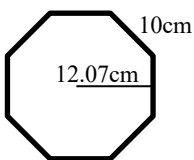
b)



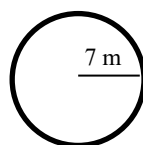
c)



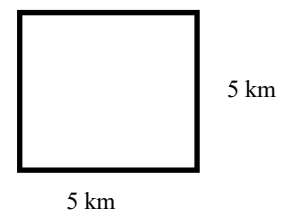
d)



e)



f)



## Geometry – 2D Shapes: Triangles

When working with triangles, we might not be given all the information necessary to determine the area or the perimeter, but we have some tools to help us determine the information we need.

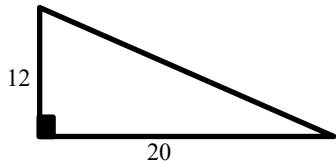
### Using the Pythagorean Theorem

If we know 2 sides of a right triangle, we can find the third side by using the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

Where  $c^2$  is always the hypotenuse (the longest side, which is across from the right angle)

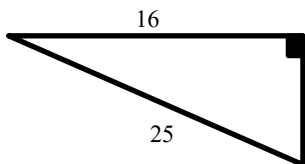
Ex: Find the missing side in the right triangle below.



$$\begin{aligned} a^2 + b^2 &= c^2 \\ 12^2 + 20^2 &= c^2 \\ 144 + 400 &= c^2 \\ 544 &= c^2 \\ 23.32 &= c \end{aligned}$$

Therefore, the missing side is 23.32 units.

Ex: Find the missing side length in the right triangle below.

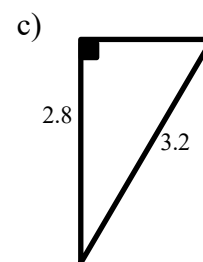
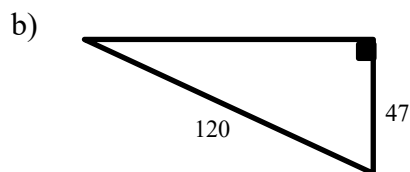
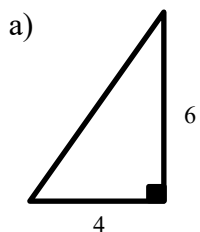


$$\begin{aligned} a^2 + b^2 &= c^2 \\ 16^2 + b^2 &= 25^2 \\ 256 + b^2 &= 625 \\ b^2 &= 369 \\ b &= 19.21 \end{aligned}$$

Therefore, the missing side is 19.21 units.

### Practice Questions

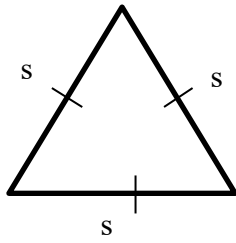
2. Find the missing side lengths in the triangles below.



## Geometry – 2D Shapes: Triangles

If we have an equilateral triangle (all the sides are the same length) or an isosceles triangle (2 sides are the same length), we can find the height of the triangle, which is useful in determining the area.

### Equilateral Triangle

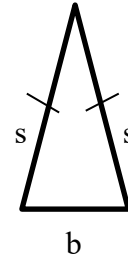


To find the height:

$$h = \sqrt{s^2 - \left(\frac{1}{2}s\right)^2}$$

where  $s$  is the length of a side

### Isosceles Triangle



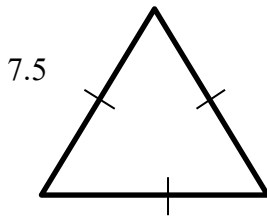
To find the height:

$$h = \sqrt{s^2 - \left(\frac{1}{2}b\right)^2}$$

where  $s$  is the length of a side  
and  $b$  is the length of the base

Ex: Find the height of the triangles below.

a)



$$h = \sqrt{s^2 - \left(\frac{1}{2}s\right)^2}$$

$$h = \sqrt{7.5^2 - \left(\frac{1}{2} \times 7.5\right)^2}$$

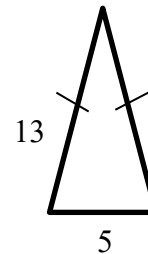
$$h = \sqrt{56.24 - (3.75)^2}$$

$$h = \sqrt{56.24 - 14.0625}$$

$$h = \sqrt{42.1775}$$

$$h = 6.49 \text{ units}$$

b)



$$h = \sqrt{s^2 - \left(\frac{1}{2}b\right)^2}$$

$$h = \sqrt{13^2 - \left(\frac{1}{2} \times 5\right)^2}$$

$$h = \sqrt{169 - (2.5)^2}$$

$$h = \sqrt{169 - 6.25}$$

$$h = \sqrt{162.75}$$

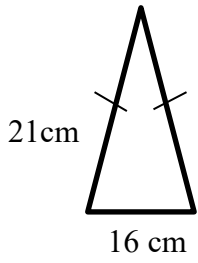
$$h = 12.757 \text{ units}$$

Geometry – 2D Shapes: Triangles

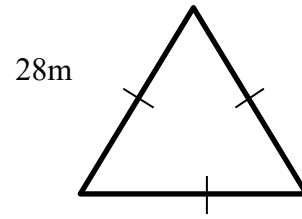
Practice Questions

3) Find the heights of the following triangles.

a)

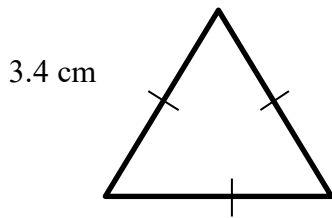


b)

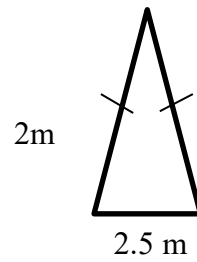


4) Find the area of the following triangles.

a)

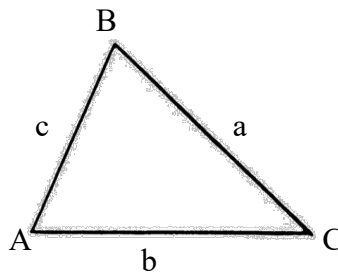


b)



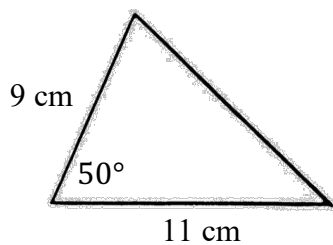
**Cosine Law**

When we need to find the missing side or missing angle of a triangle that is not a right-angled triangle, we can use **cosine law**.



To find a side length	To find an angle
$a^2 = b^2 + c^2 - 2bc \cos A$ $b^2 = a^2 + c^2 - 2ac \cos B$ $c^2 = a^2 + b^2 - 2ab \cos C$	$A = \cos^{-1} \left( \frac{a^2 - b^2 - c^2}{-2bc} \right)$ $B = \cos^{-1} \left( \frac{b^2 - a^2 - c^2}{-2ac} \right)$ $C = \cos^{-1} \left( \frac{c^2 - a^2 - b^2}{-2ab} \right)$

Ex: Find the missing side of the triangle below.



$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 9^2 + 11^2 - 2 \times 9 \times 11 \times \cos 50$$

$$a^2 = 81 + 121 - 198 \times 0.6428$$

$$a^2 = 202 - 127.2744$$

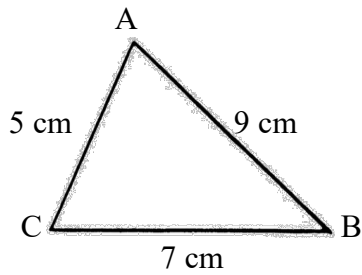
$$a^2 = 74.7256$$

$$a = \sqrt{74.7256}$$

$$a = 8.64 \text{ cm}$$

## Geometry – 2D Shapes: Triangles

Ex: Find the missing angles of the triangle below.



$$A = \cos^{-1} \left( \frac{a^2 - b^2 - c^2}{-2bc} \right)$$

$$A = \cos^{-1} \left( \frac{7^2 - 5^2 - 9^2}{-2 \times 5 \times 9} \right)$$

$$A = \cos^{-1} \left( \frac{49 - 25 - 81}{-2 \times 5 \times 9} \right)$$

$$A = \cos^{-1} \left( \frac{-57}{-90} \right)$$

$$A = 50.71^\circ$$

$$B = \cos^{-1} \left( \frac{b^2 - a^2 - c^2}{-2ac} \right)$$

$$B = \cos^{-1} \left( \frac{5^2 - 7^2 - 9^2}{-2 \times 7 \times 9} \right)$$

$$B = \cos^{-1} (0.8333)$$

$$B = 33.58^\circ$$

$$C = 180 - 50.71 - 33.58$$

$$C = 95.71^\circ$$

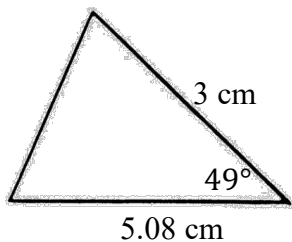


Geometry – 2D Shapes: Triangles

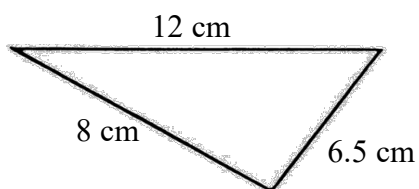
Practice Questions

5) Calculate the unknown measurements in each triangle below.

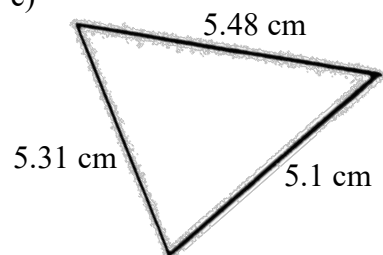
a)



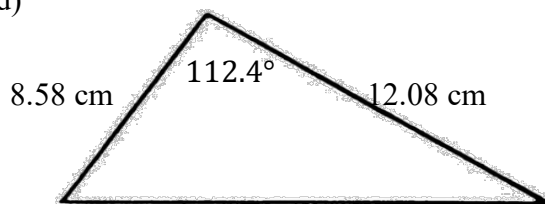
b)



c)



d)



## Geometry – 2D Shapes: Apothem

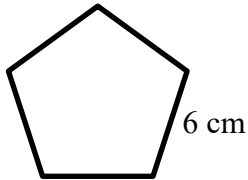
### Finding apothem

If we have a regular “-agon” and we are not given the measurement of the apothem, we can find it if we know the side length.

$$Apothem = \frac{side\ length \div 2}{\tan((360 \div \#\ of\ sides) \div 2)}$$

Ex: Find the length of the apothem in each of the regular polygons below.

a)



$$Apothem = \frac{side\ length \div 2}{\tan((360 \div \#\ of\ sides) \div 2)}$$

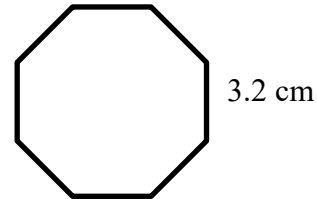
$$Apothem = \frac{6 \div 2}{\tan((360 \div 5) \div 2)}$$

$$Apothem = \frac{3}{\tan 36}$$

$$Apothem = \frac{3}{0.7265}$$

$$Apothem = 4.13\ cm$$

b)



$$Apothem = \frac{side\ length \div 2}{\tan((360 \div \#\ of\ sides) \div 2)}$$

$$Apothem = \frac{3.2 \div 2}{\tan((360 \div 8) \div 2)}$$

$$Apothem = \frac{1.6}{\tan 22.5}$$

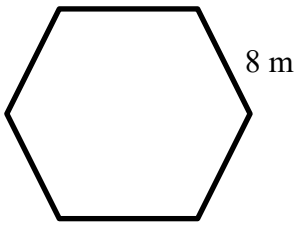
$$Apothem = \frac{1.6}{0.4142}$$

$$Apothem = 3.86\ cm$$

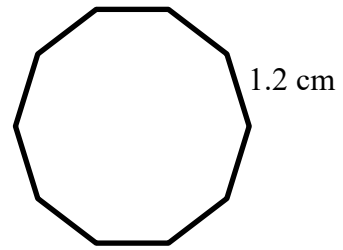
Geometry – 2D Shapes: Apothem  
Practice Questions

6) Find the apothem of the following regular polygons.

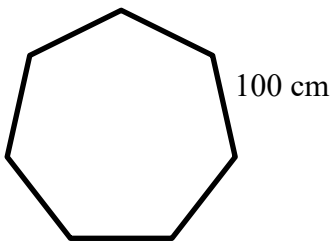
a)



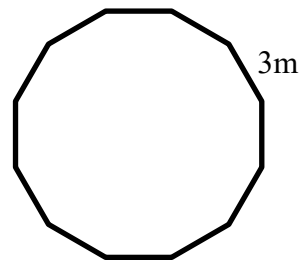
b)



c)



d)



## Geometry – 2D Shapes: Missing Measures

### Missing Measures

Instead of using the formulas we've just been working with to find area, perimeter, height, or apothem, we can use the same formulas to find missing measures.

To do this, we need to do the following:

- 1) Choose a formula based on what we know and what we want to find.
- 2) Plug in what we know.
- 3) Use algebra to solve for the missing variable.

Ex: A triangle has an area of  $12 \text{ cm}^2$  and a base of  $2 \text{ cm}$ . What is the height of the triangle?

We know area and want height so we should start with the area formula that includes height.

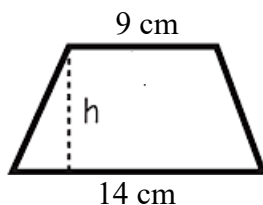
$$A = \frac{b \times h}{2}$$

$$12 = \frac{2 \times h}{2}$$

$$24 = 2 \times h$$

$$12 \text{ cm} = h$$

Ex: What is the height of the trapezoid below?



$$A = 92 \text{ cm}^2$$

$$A = \frac{(b_1 + b_2) \times h}{2}$$

$$92 = \frac{(9 + 14) \times h}{2}$$

$$184 = 23 \times h$$

$$h = 8 \text{ cm}$$

## Geometry – 2D Shapes: Missing Measures

### Practice Questions

7) Find the missing measures of the shapes below.

a) A circular table has a circumference of 24.93 ft

What is the radius?

What is the diameter?

What is the area?

b) A circular dartboard has an area of  $333.12 \text{ in}^2$ .

What is the radius?

What is the diameter?

What is the circumference?

c) An isosceles triangle has a height of 12 m and an area of  $48 \text{ m}^2$ .

What is the base of the triangle?

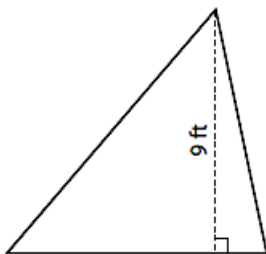
What is the perimeter of the triangle?

d) An isosceles triangle has a base of 16 m and an area of  $176 \text{ m}^2$ .

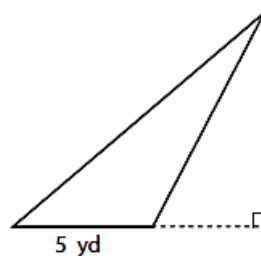
What is the height of the triangle?

What is the perimeter of the triangle?

e) What is the base of the triangle shown below if the area is  $36 \text{ ft}^2$ ?



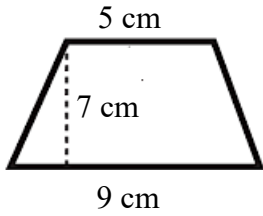
f) What is the height of the triangle shown below if the area is  $25 \text{ yd}^2$ ?



Geometry – 2D Shapes: Equivalent Figures  
**Equivalent Figures**

Two figures are considered equivalent if they have the same area.

Ex: Show that the following figures are equivalent.



Area of trapezoid

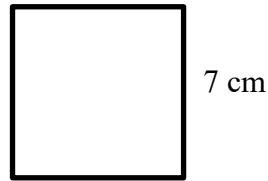
$$A = \frac{(b_1 + b_2) \times h}{2}$$

$$A = \frac{(5 + 9) \times 7}{2}$$

$$A = \frac{14 \times 7}{2}$$

$$A = \frac{(b_1 + b_2) \times h}{2}$$

$$A = 49 \text{ cm}^2$$



Area of square

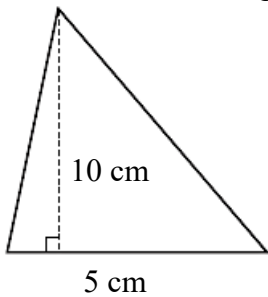
$$A = s^2$$

$$A = 7^2$$

$$A = 49 \text{ cm}^2$$

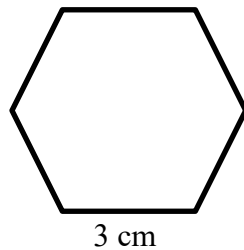
Because both figures have the same area, they are equivalent.

Ex: Are the following figures equivalent?



$$A = \frac{b \times h}{2}$$

$$A = 25 \text{ cm}^2$$



$$\text{Apothem} = \frac{\text{side length} \div 2}{\tan((360 \div \# \text{ of sides}) \div 2)}$$

$$\text{Apothem} = 2.5981$$

$$A = \frac{P \times a}{2}$$

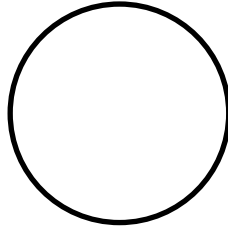
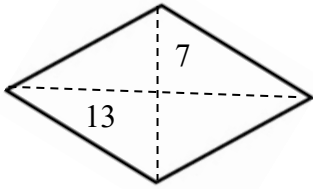
$$A = \frac{18 \times 2.5981}{2}$$

$$A = 23.38 \text{ cm}^2$$

The figures are not equivalent.

Geometry – 2D Shapes: Equivalent Figures

Ex: Given that the following figures are equivalent, what is the radius of the circle?



Area of rhombus

$$A = \frac{d_1 \times d_2}{2}$$

$$A = \frac{13 \times 7}{2}$$

$$A = 45.5 \text{ units}^2$$

Radius of circle

$$A = \pi r^2$$

$$45.5 = \pi \times r^2$$

$$14.4831 = r^2$$

$$r = 3.81 \text{ units}$$

The radius is 3.81 units

Practice Questions

8) A triangle has a base of 9 cm and a height of 8 cm. What is the side length of a square that is equivalent to the triangle?

9) In each of the following cases, find the measure  $x$  of the side of the square equivalent to:

a. An 8 cm by 12 cm rectangle:

b. A right triangle with sides measuring 3 cm, 6 cm and 6.7 cm:

c. A trapezoid with a big base of 12 cm, a small base of 4 cm and a height of 8 cm:

d. A rhombus with diagonals measuring 5 cm and 10 cm.

## Geometry – 2D Shapes: Rules of Equivalency

### Rules of Equivalency

Remember that a polygon is considered “regular” if all the sides have the same length.

There are a couple of rules around equivalency:

- 1) If you have equivalent polygons with the same number of sides, the regular polygon will always have the smallest perimeter.
- 2) If you have polygons with the same number of sides and the same perimeter, the regular polygon will always have the greatest area.
- 3) If you have equivalent polygons, the one with the greatest number of sides will always have the smallest perimeter.
- 4) If you have polygons with the same perimeter, the one with the most number of sides will always have the greatest area.

Ex: A regular triangle, square, and pentagon all have the same perimeters. Which one has the largest area?

Using Rule #4, if we have polygons with the same perimeters, the polygon with the most number of sides has the greatest area. Therefore, the pentagon has the greatest area.

Ex: Using 100 m of fence, a farmer wants to make a rectangular enclosure for a herd of sheep. What is the maximum area of this enclosure?

From Rule #2, we know that the rectangle must be regular (so it must be a square).  
A square with a perimeter of 100 m has side lengths of 25m.

$$A = 25^2$$
$$A = 625 \text{ m}^2$$



## Geometry – 2D Shapes: Rules of Equivalency

### Practice Questions

10) An equilateral, isosceles and a scalene triangle all have the same perimeter. Which one has the greatest area?

11) A regular pentagon and hexagon are equivalent. Which one has the smallest perimeter?

12) A regular pentagon and hexagon have the same perimeter. Which one has the greatest area?

13) What is the maximum area of a rectangle with a perimeter of 16 cm?

14) What is the maximum area, to the nearest hundredth, of a triangle with perimeter of 12cm?

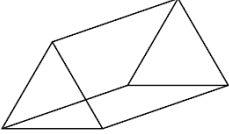
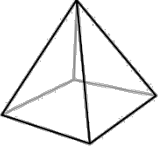
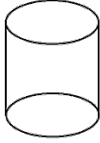
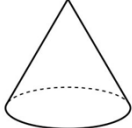
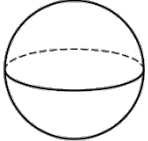
Geometry – 3D Solids: Area and Volume

**Area and Volume**

When considering 3-dimensional solids, we can talk about area (surface and lateral) and volume.

We can think of area as the amount of wrapping paper it would take to cover the object. For surface area we want to cover all the surfaces. For lateral area we leave the top and bottom uncovered.

We can think of volume as the amount of space inside the object.

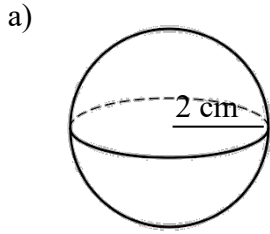
Solid	Lateral Area	Surface Area	Volume
<p><b>Prisms</b></p> <p>2 bases that can be any shape except circles</p> 	$A_l = P_b \times h$	$A_s = 2A_b + A_l$	$V = A_b \times h$
<p><b>Pyramids</b></p> <p>1 base that can be any shape except a circle</p> 	$A_l = \frac{P_b \times s}{2}$	$A_s = A_b + A_l$	$V = \frac{A_b \times h}{3}$
<p><b>Cylinders</b></p> <p>Like a prism with a circular base</p> 	$A_l = 2\pi r h$	$A_s = 2A_b + A_l$	$V = A_b \times h$
<p><b>Cones</b></p> <p>Like a pyramid with a circular base</p> 	$A_l = 2\pi r s$	$A_s = A_b + A_l$	$V = \frac{A_b \times h}{3}$
<p><b>Spheres</b></p> 	<p>None</p>	$A_s = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$

$A_l$ : Lateral area     $A_s$ : Surface area     $A_b$ : Area of base     $P_b$ : Perimeter of base

$s$ : Slant height     $h$ : Height     $r$ : Radius

Geometry – 3D Solids: Area and Volume

Ex: Determine the lateral area, surface area, and volume of each of the following figures.



Lateral Area  
None

Surface Area  
 $A_s = 4\pi r^2$

Volume  
 $V = \frac{4}{3}\pi r^3$

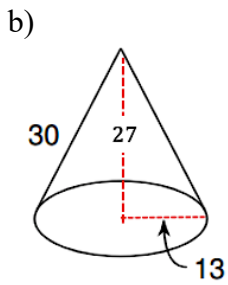
$$A_s = 4\pi 2^2$$

$$V = \frac{4}{3}\pi \times 2^3$$

$$A_s = 4\pi \times 4$$

$$A_s = 50.27 \text{ cm}^2$$

$$V = 33.51 \text{ cm}^3$$



Lateral Area  
 $A_l = 2\pi rs$

Surface Area  
 $A_s = A_b + A_l$

Volume  
 $V = \frac{A_b \times h}{3}$

$$A_l = 2\pi \times 13 \times 30$$

$$A_s = \pi r^2 + 2450$$

$$V = \frac{\pi r^2 \times 27}{3}$$

$$A_l = 2450.44 \text{ units}^2$$

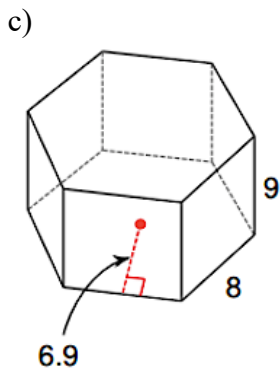
$$A_s = \pi \times 13^2 + 2450$$

$$V = \frac{\pi \times 13^2 \times 27}{3}$$

$$A_s = \pi r^2 + 2450$$

$$A_s = 2981.37 \text{ units}^2$$

$$V = 4778.36 \text{ units}^3$$



Lateral Area  
 $A_l = P_b \times h$

Surface Area  
 $A_s = 2A_b + A_l$

Volume  
 $V = \left(\frac{P \times a}{2}\right) \times 9$

$$A_l = 8 \times 6 \times 9$$

$$A_s = 2\left(\frac{P \times a}{2}\right) + 432$$

$$V = \left(\frac{48 \times 6.9}{2}\right) \times 9$$

$$A_l = 432 \text{ units}^2$$

$$A_s = 2\left(\frac{48 \times 6.9}{2}\right) + 432$$

$$V = 1490.4 \text{ units}^3$$

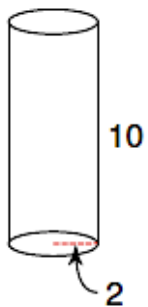
$$A_s = 763.2 \text{ units}^2$$

Geometry – 3D Solids: Area and Volume

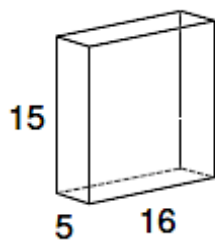
Practice Questions

15) Determine the lateral area, surface area, and volume of each of the following figures.

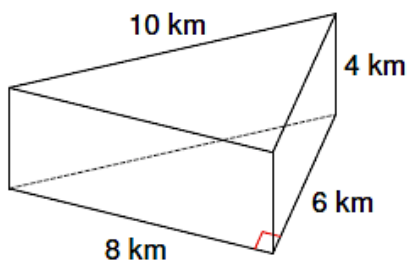
a)



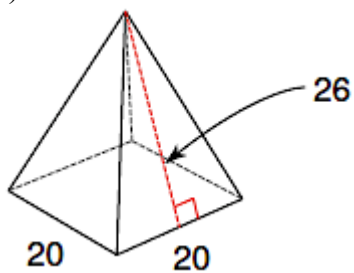
b)



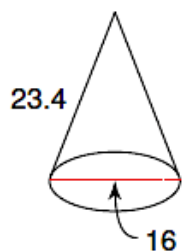
c)



d)



e)



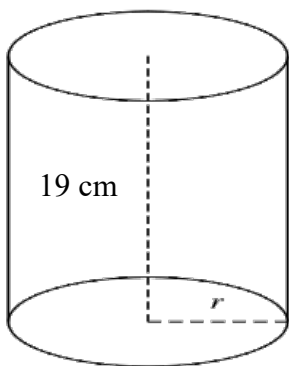
### Missing Measures

Just like with 2-dimensional figures, we can use the formulas we've been working with to find lateral area, surface area, and volume to find missing measures.

To do this, we need to do the following:

- 1) Choose a formula based on what we know and what we want to find.
- 2) Plug in what we know.
- 3) Use algebra to solve for the missing variable.

Ex: The cylinder below has a volume of  $238.761 \text{ cm}^3$ . What is the radius?



$$V = A_b \times h$$
$$238.761 = \pi \times r^2 \times 19$$

$$4 = r^2$$

$$2 \text{ cm} = r$$

Ex: A square-based pyramid has a surface area of  $172.8 \text{ cm}^2$ . The square base has sides measuring 6 cm. What is the slant height?

$$A_s = A_b + A_l$$

$$172.8 = 6 \times 6 + \frac{24 \times s}{2}$$

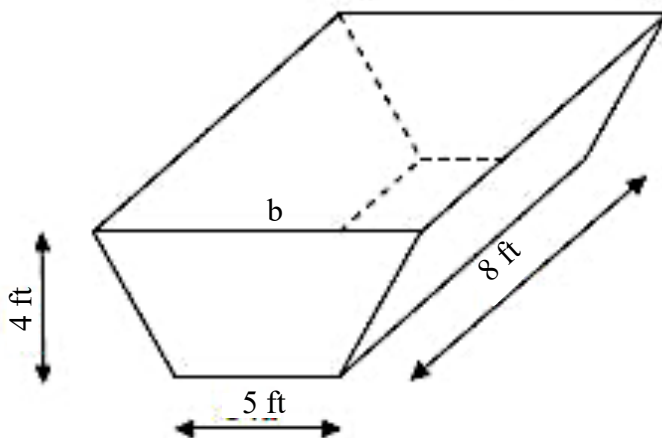
$$172.8 = 36 + 12 \times s$$

$$11.4 \text{ cm} = s$$

Geometry – 3D Solids: Missing Measures

Practice Questions

16) Find the missing base,  $b$ , given that the volume is  $176 \text{ ft}^3$ .



17) A cylinder has a volume of approximately  $314 \text{ cm}^3$  and a radius of 5 cm. Solve for the height of the cylinder.

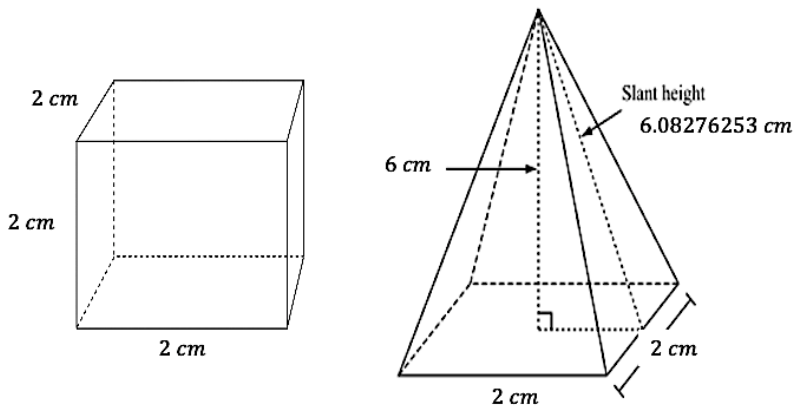
18) A cone has a volume of  $37.68 \text{ mm}^3$  and a height of 9 mm. Find the radius.

## Geometry – 3D Solids: Equivalent Solids

### Equivalent Solids

Solids are considered equivalent if they have the same volume. The shape, lateral area, and surface area does not matter.

Ex: Are the two following solids equivalent?



Volume Cube

$$V = s^3$$

$$V = 2^3$$

$$V = 8 \text{ cm}^3$$

Volume Pyramid

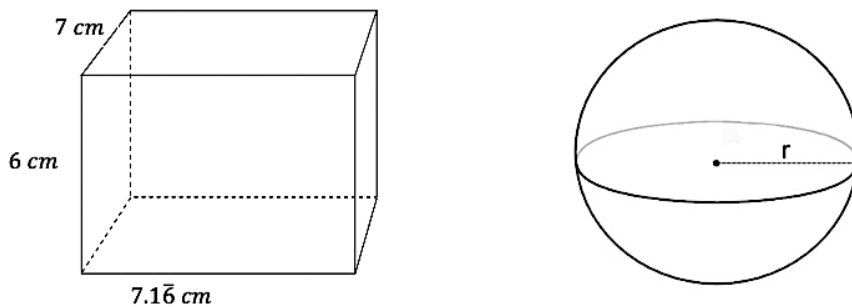
$$V = \frac{A_b \times h}{3}$$

$$V = \frac{4 \times 6}{3}$$

$$V = 8 \text{ cm}^3$$

Yes, the solids are equivalent.

Ex: Given that the two following solids are equivalent, what is the radius of the sphere?



Volume of Rectangular Prism

$$V = A_b \times h$$

$$V = 7.167 \times 6 \times 7$$

$$V = 301.014 \text{ cm}^3$$

Radius of Sphere

$$V = \frac{4}{3} \pi r^3$$

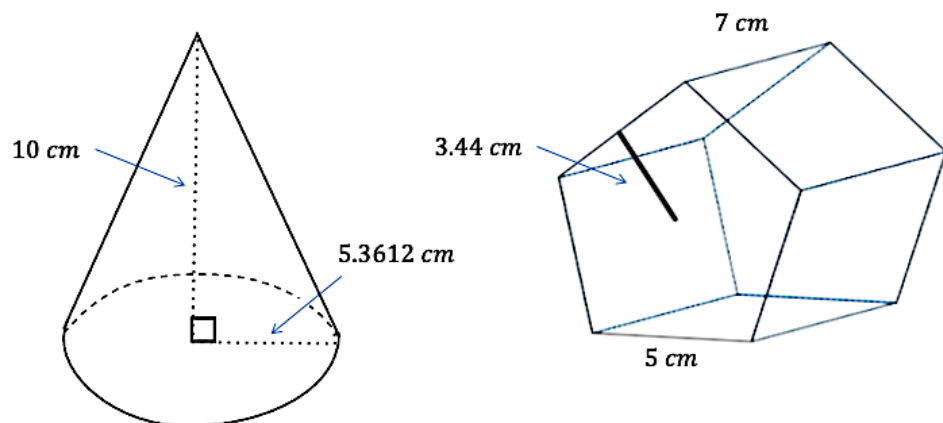
$$301.014 = \frac{4}{3} \pi r^3$$

$$71.86 = r^3$$

$$4.16 \text{ cm} = r$$

Geometry – 3D Solids: Equivalent Solids  
Practice Questions

19) Are the two solids equivalent?



20) A prism with a height of 4 cm has a rectangular base with dimensions 6cm by 9cm. What is the measure of a cube's edge that is equivalent to the prism?

21) A cone and a cylinder are equivalent. The radius and the height of the cone measure 6 cm and 10 cm respectively. What is the height of the cylinder if its radius measures 5 cm?



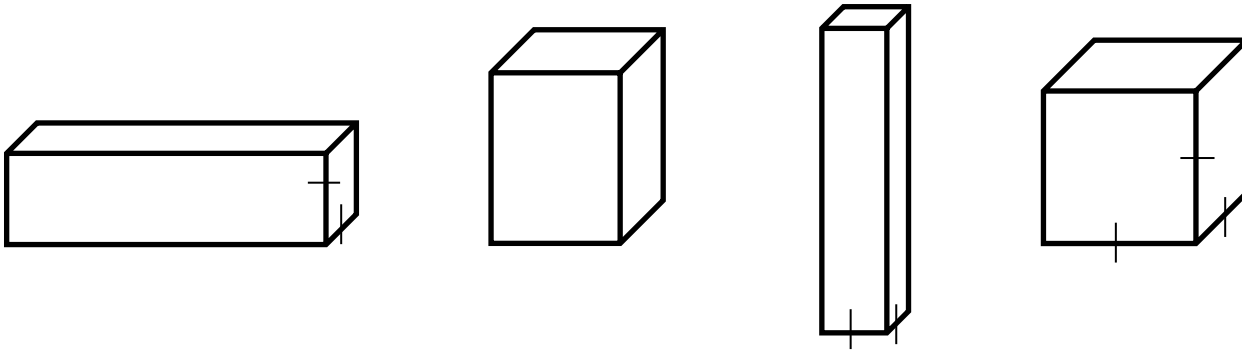
## Geometry – 3D Solids: Rules of Equivalency

### Rules of Equivalency

Just like we have rules of equivalency for two-dimensional shapes, there are rules of equivalency for three-dimensional solids.

- 1) If you have equivalent solids with the same number of faces, the regular solid will always have the smallest surface area.
- 2) If you have solids with the same number of faces and the same surface area, the regular solid will have the greatest volume.
- 3) If you have equivalent solids, the one with the greatest number of faces will always have the smallest surface area.
- 4) If you have solids with the same surface area, the one with the most faces will have the greatest volume.

Ex: Given the following solids have the same number of faces and the same surface area, which one will have the greatest volume?



Ex: Of all solids with a total volume of  $8 \text{ cm}^3$ , what is the shape of the solid with the smallest total area?

Geometry – 3D Solids: Rules of Equivalency

Practice Questions

22) What is the maximum volume of a rectangular prism with a total surface area of  $150 \text{ cm}^2$ ?

23) What is the minimum surface area of a rectangular prism with a total volume of  $512 \text{ cm}^3$ ?

24) What is the minimum area of a solid with a total volume of  $113 \text{ cm}^3$ ?

Answer Key – Financial Math

- 1) a) 512      b) 0.008      c) 4      d) 5      e) 2      f) 1      g) 0      h) 1.025
- 2) a) True      b) False      c) True      d) True      e) False      f) True
- 3) a)  $x = 2$       b)  $x = \frac{1}{3}$       c)  $x = 4$       d)  $x = \frac{1}{3}$       e)  $x = 6$       f)  $x = 2$       g)  $x = \frac{1}{2}$       h)  $x = -4$
- 4) 143 residents
- 5) 206 species
- 6) 16 384 bacteria
- 7) 50%
- 8) 12%
- 9) 389.7 cm
- 10) a)  $x = \log_{16} 8$       b)  $x = \log_{1.2} 0.58$       c)  $5 = \log_{1.4} 5.37824$
- 11) a)  $x = 2.73$       b)  $x = 0.64$       c)  $x = 162.1$
- 12) 6.6 hours
- 13) 10.5 hours
- 14) \$10 800
- 15) \$3713.40
- 16) Yes
- 17) \$2675
- 18) 40 months
- 19) 13.2%
- 20) 1460 days
- 21) \$9856.56
- 22) 2.85%
- 23) \$6691.13

Answer Key – Financial Math

24) \$222.23

25) \$4314.75

26) \$3000.03

27) 6 years

28) 5%

29) 5 years

30) 7.177%

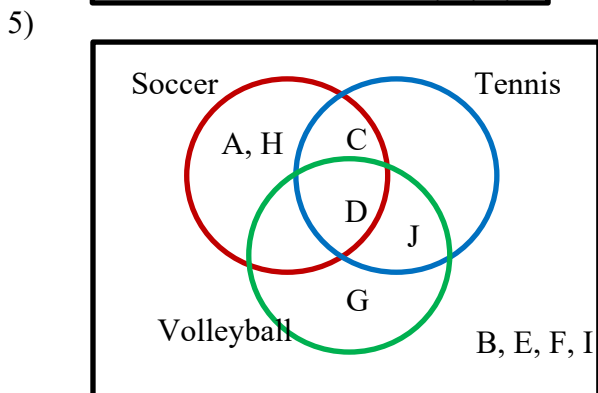
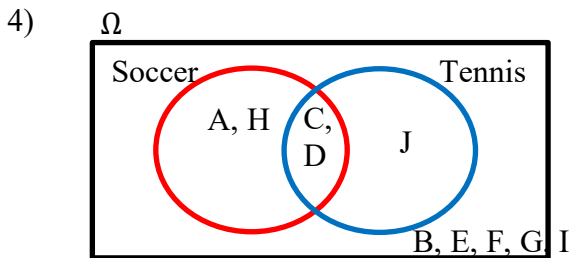
31) 6 years

Answer Key – Probability

1)  $P = \frac{15}{100}$

2) a)  $P = \frac{4}{52}$  or  $\frac{1}{13}$       b)  $P = \frac{26}{52}$  or  $\frac{1}{2}$       c)  $P = \frac{13}{52}$  or  $\frac{1}{4}$       d)  $P = \frac{1}{52}$

3) a)  $P = \frac{4}{15}$       b)  $P = \frac{2}{15}$       c)  $P = \frac{6}{15}$       d)  $P = \frac{3}{15}$  or  $\frac{1}{5}$



6)  $P = \frac{2}{5}$

7) a) No. The possible outcomes are: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. We can see that 8, 10, and 12 are greater than or equal to 7 and even (satisfying both A and B), so A and B are not mutually exclusive.

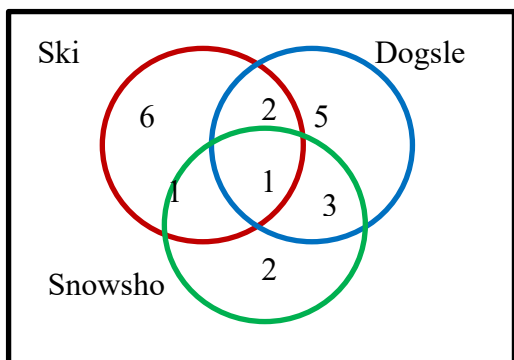
b) i)  $P = \frac{6}{11}$       ii)  $P = \frac{5}{11}$

8) a)  $P = \frac{1}{6}$       b)  $P = \frac{4}{6}$  or  $\frac{2}{3}$       c)  $P = \frac{2}{6}$  or  $\frac{1}{3}$       d)  $P = \frac{2}{6}$  or  $\frac{1}{3}$

9) a)  $P = \frac{7}{16}$       b)  $P = \frac{9}{16}$       c)  $P = \frac{4}{16}$  or  $\frac{1}{4}$       d)  $P = \frac{11}{16}$

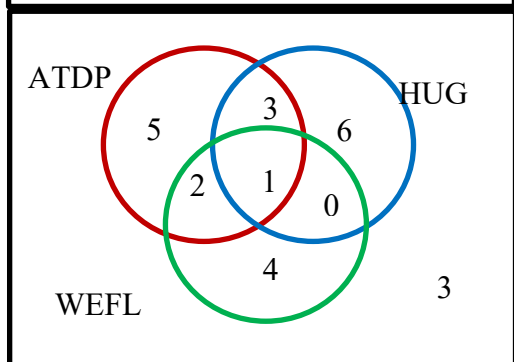
Answer Key – Probability

10) a)



b)  $P = \frac{1}{20}$

11) a)



b)  $P = \frac{3}{24}$  or  $\frac{1}{8}$

12)

a)  $P = \frac{12}{115}$

b)  $P = \frac{23}{115}$  or  $\frac{1}{5}$

c)  $P = \frac{85}{115}$  or  $\frac{17}{23}$

13)

a)  $P = \frac{12}{20}$  or  $\frac{3}{5}$

b)  $P = \frac{12}{30}$  or  $\frac{2}{5}$

c)  $P = \frac{8}{10}$  or  $\frac{4}{5}$

d)  $P = \frac{18}{20}$  or  $\frac{9}{10}$

14)

a)  $P = \frac{4}{13}$

b)  $P = \frac{16}{31}$

c)  $P = \frac{13}{46}$

d)  $P = \frac{13}{31}$

e)  $P = 0$

f)  $P = 0$

15)

a)  $P = \frac{12}{30}$  or  $\frac{2}{5}$

b)  $P = \frac{12}{23}$

c)  $P = \frac{17}{41}$

d)  $P = \frac{7}{29}$

16) a)

	Home Economics	Biology	Law	Total
Art	15	25	80	120
Music	25	15	10	50

Answer Key – Probability

Drama	20	10	0	30
Total	60	50	90	200

b)  $P = \frac{50}{200}$  or  $\frac{1}{4}$

c)  $P = \frac{20}{200}$  or  $\frac{1}{10}$

d)  $P = \frac{25}{50}$  or  $\frac{1}{2}$

e)  $P = \frac{20}{30}$  or  $\frac{2}{3}$

17)

a)  $P = \frac{9}{50}$

b)  $P = \frac{1}{6}$

c)  $P = \frac{11}{120}$

d)  $P = \frac{9}{250}$

18) 4096

19) 3 603 600

20) a) 420 possibilities

b)  $P = \frac{1}{420}$

c)  $P = \frac{5}{420} = \frac{1}{84}$

21) a) 45

b)  $P = \frac{9}{45} = \frac{1}{5}$

22) a) 2: 5 or  $\frac{2}{5}$

b) 5: 2 or  $\frac{5}{2}$

23) a) 6: 30 or 1: 5 or  $\frac{6}{30}$  or  $\frac{1}{5}$

b) 35: 1 or  $\frac{35}{1}$

c) 1: 35 or  $\frac{1}{35}$

d) 0

24)  $E = -\$0.40$  (or an expected loss of \$0.40)

25)  $E = \$1.50$  (or an expected gain of \$1.50)

26) The game is not fair

27) a) The game is not fair

b) To be fair, the game should cost \$2.40 to play

28) 71.9%

29) Car B

## Answer Key – Voting

1) Candidate B

2) Tacos

3) Village A

4) No winner

5) Pizza

6) Candidate C

7) Blue

8) Vietnamese

9) Pizza

10) Blue

11) Vietnamese

12) a) Liberal: 114 seats; Conservative: 119 seats; BQ: 26 seats; NDP: 55 seats; Green: 23 seats; Independent: 1 seat

b) Liberals would lose 43 seats; Conservatives would lose 2 seats; BQ would lose 6 seats; NDP would gain 31 seats; Green would gain 20 seats; Independents would stay the same

13) Sushi

14) Tacos



Answer Key – Geometry

1)

a)  $A = 150 \text{ cm}^2$   
 $P = 53.31 \text{ cm}$

b)  $A = 28 \text{ cm}^2$   
 $P = 22 \text{ cm}$

c)  $A = 96 \text{ m}^2$   
 $P = 50 \text{ m}$

d)  $A = 482.8 \text{ cm}^2$   
 $P = 80 \text{ cm}$

e)  $A = 153.94 \text{ m}^2$   
 $P = 43.98 \text{ cm}$

f)  $A = 25 \text{ km}^2$   
 $P = 20 \text{ cm}$

2)

a) 7.21 units

b) 110.41 units

c) 1.55 units

3)

a) 19.42 cm

b) 24.25 cm

4)

a)  $5 \text{ cm}^2$

b) 1.95

5)

a) side: 8.85 cm  
 angle:  $36^\circ$   
 angle:  $95^\circ$

b) angle:  $38.4^\circ$   
 angle:  $30.3^\circ$   
 angle:  $111.3^\circ$

c) angle:  $56.4^\circ$   
 angle:  $60.1^\circ$   
 angle:  $63.5^\circ$

d) side: 17.28 cm  
 angle:  $40.3^\circ$   
 angle:  $27.3^\circ$

6)

a) 6.9 m

b) 1.85 m

c) 103.8 cm

d) 5.6 m

7)

a)  $r = 3.97 \text{ ft}$   
 $d = 7.94 \text{ ft}$   
 $A = 49.51 \text{ ft}^2$

b)  $r = 10.3 \text{ in}$   
 $d = 20.6 \text{ in}$   
 $A = 64.7 \text{ in}$

c)  $b = 8 \text{ m}$   
 $p = 33.3 \text{ m}$

d)  $h = 22 \text{ m}$   
 $p = 48 \text{ m}$

e)

$b = 8 \text{ ft}$

f)

$h = 10 \text{ yd}$

8)  $s = 6 \text{ cm}$

9)

a) 9.8 cm

b) 3 cm

c) 8 cm

d) 5 cm

10) Equilateral triangle

11) Hexagon

12) Hexagon

13)  $16 \text{ cm}^2$

14)  $6.92 \text{ cm}^2$

Answer Key – Geometry

15)

a)

$$A_l = 125.7 \text{ units}^2$$

$$A_s = 150.8 \text{ units}^2$$

$$V = 125.7 \text{ units}^3$$

d)

$$A_l = 1040 \text{ units}^2$$

$$A_s = 1440 \text{ units}^2$$

$$V = 3200 \text{ units}^3$$

b)

$$A_l = 630 \text{ units}^2$$

$$A_s = 790 \text{ units}^2$$

$$V = 1200 \text{ units}^3$$

e)

$$A_l = 1176.2 \text{ units}^2$$

$$A_s = 1377.3 \text{ units}^2$$

$$V = 1473.8 \text{ units}^3$$

c)

$$A_l = 96 \text{ km}^2$$

$$A_s = 144 \text{ km}^2$$

$$V = 96 \text{ km}^3$$

16)  $b = 6 \text{ ft}$

17)  $h = 4 \text{ cm}$

18)  $2\text{mm}$

19) Yes

20)  $6 \text{ cm}$

21)  $7.2 \text{ cm}$

22)  $125 \text{ cm}^3$

23)  $384 \text{ cm}^2$

24)  $113 \text{ cm}^3$