Math Workbook
Grade 11 CST
2021-22

Optimization *Graph Theory*

Financial Math *Voting*

Probability *Geometry*

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Math Workbook
Grade 11 CST
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NAME___________________________________

SECTION_________________________________

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Review Unit – Solving Algebraic Equations

R.1 SOLVING ALGEBRAIC EQUATIONS

Variables are letters or symbols that represent numbers we do not know or that can change. We can use algebra to find the exact value of the variable.

There are two keys to solving equations:

1) To get rid of a number, you do its opposite (for example, if it was addition, use subtraction).
2) Whatever you do to one side, you do the same to the other side.

Let's look at 5 different levels.

Level 1: Variable on one side with only addition or subtraction

Get rid of the addition or subtraction by doing the opposite on both sides.

Ex: Solve for x.

a) $x + 2 = 3$ b) $x - 4 = 8$ c) $4x = 20$ d) $\frac{x}{10} = 3$

Practice Questions

1) Solve for x:

a) $x - 6 = 10$ b) $-8 - x = 11$ c) $45 = 5x$ d) $18 = \frac{x}{4}$

Review Unit – Solving Algebraic Equations

Level 2: Variable on one side with addition or subtraction and multiplication.

Get rid of the addition or subtraction first, then get rid of the multiplication or division.

Ex: Solve for x.

a) $3x + 9 = 12$

b) $4 + 2x = 10$

c) $\frac{x}{3} - 25 = 75$

d) $-7 + \frac{x}{5} = 20$

Practice Questions

2) Solve for x:

a) $4x - 6 = 6$

b) $8 + \frac{x}{7} = 43$

c) $15 = -5 + \frac{x}{2}$

d) $33 = 10x + 3$

Review Unit – Solving Algebraic Equations

Level 3: Variables and addition or subtraction on both sides

Get rid of the variable on one side, then get rid of the addition or subtraction on the other side.

Ex: Solve for x.

a) $3x + 3 = 13 - 2x$

b) $4 - 4x = 18 - 6x$

c) $28x - 15 = 8x + 65$

d) $150 + 2x = 300 - x$

Practice Questions

3) Solve for x:

a) $2x - 6 = 8 - 5x$

b) $8 + 7x = 3x + 28$

c) $15 - 2x = -5 + 2x$

d) $20x + 60 = -10x + 120$

Review Unit – Solving Algebraic Equations

Level 4: Brackets on one or both sides

Get rid of the brackets by expanding and then solve like in Level 3.

Ex: Solve for x.

a) $6(x-2)=15+3x$ b) $3(4+3x)=4(2+x)$ c) $2(7-8x)=5(2x-4)$

Practice Questions

4) Solve for x:

a) $2(x+3)=10-2x$ b) $7(2x+3)=3(4x-2)$ c) $3(2-5x)=5(4-6x)$

Review Unit – Solving Algebraic Equations

Level 5: Division (fractions) on one or both sides

Cross multiply then solve like Level 4.

Ex: Solve for x.

a) $\frac{3x+2}{4} = \frac{2-6x}{5}$

b) $\frac{4-x}{3} = \frac{5-2x}{2}$

c) $\frac{8x+3}{2} = 2 + 3x$

Practice Questions

5) Solve for x:

a) $\frac{2+4x}{3} = \frac{3x-1}{2}$

b) $\frac{2x+7}{2} = \frac{4-6x}{4}$

c) $8x + 3 = \frac{12x-5}{5}$

Review Unit – Solving Algebraic Equations

Practice Questions

6) $x + 3 = 5$

7) $2x + 4 = x + 12$

8) $3(x + 4) = 2(5 - 2x)$

9) $\frac{2x-3}{4} = \frac{2+x}{5}$

10) $3x - 6 = 18$

11) $x + 1 = 4(2 + 4x)$

12) $\frac{18x+3}{6} = 5x - 2$

13) $4x - 3 = 13$

Review Unit – Linear Equations
R.2 LINEAR EQUATIONS

Linear equations are generally written in the form $y = ax + b$

a is the slope (or rate of change)

b is the y-intercept (or initial value)

Identify the slope and y-intercept in the following equations

Ex: Find the slope and y-intercept of the following lines.

a) $y = 3x + 4$	b) $y = \frac{2}{3}x - 7$	c) $y = x + 10$	d) $y = -\frac{4}{3}x - 12$
slope:	slope:	slope:	slope:
y-intercept:	y-intercept:	y-intercept:	y-intercept:

Sometimes we need to re-arrange the equation to get it in the form $y = ax + b$ before we can identify the slope and the y-intercept.

Ex: Find the slope and y-intercept of the following lines.

a) $2y = 8x + 4$	b) $4y - 5x = 12$
slope:	slope:
y-intercept:	y-intercept:

Practice Questions

1) Identify the slope and y-intercept of the following linear equation

a) $3x + 4y = 12$

b) $-2y - 10x + 4 = 0$

Review Unit – Linear Equations

Graphing a line

When we are graphing a line:

$$a = slope = \frac{rise}{run}$$

b = y-intercept (where the line crosses the y-axis).

Positive slope = "rise" up.
Negative slope = "rise" down.
Always run right.

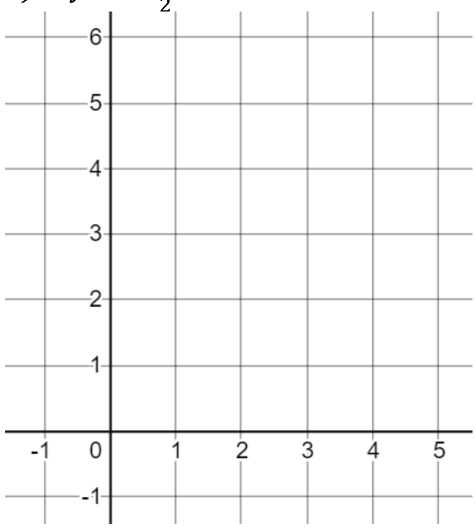
To graph a line:

1) Put a dot on the y-axis at b.
2) Starting from b, use slope to find a second point. Put a dot there.
3) Connect the dots using a ruler.
4) Draw arrows at each end of the line.

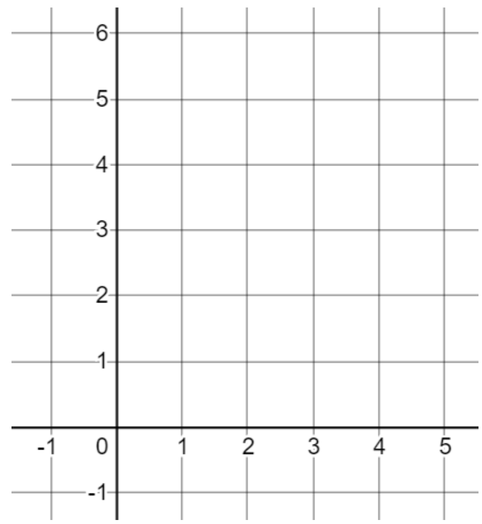
Remember:
If there is no b,
then $b = 0$

Ex: Graph the following lines.

a) $y = -\frac{3}{2}x + 4$



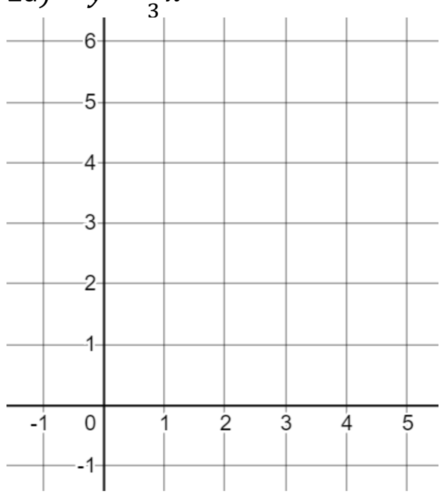
b) $y = 3x + 2$



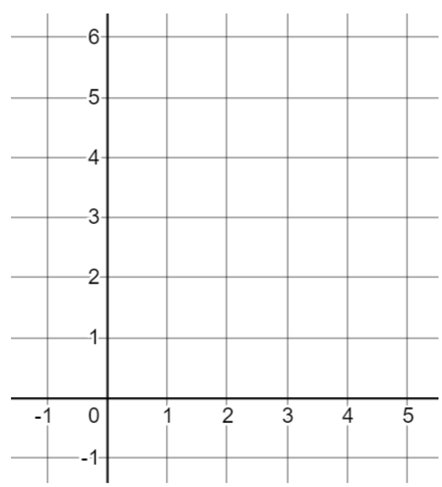
Practice Questions

2) Graph the following lines.

2a) $y = \frac{4}{3}x$



b) $y = -2x + 5$



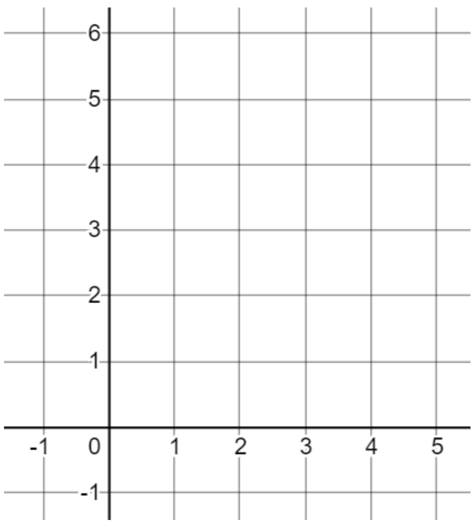
Review Unit – Linear Equations

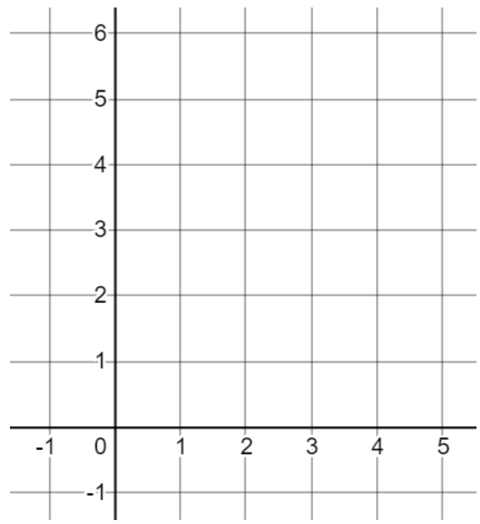
Sometimes we will have to re-arrange the equations before we can graph them.

Ex: Graph the following lines.

a) $x + y = 5$

b) $2x = 3y - 3$



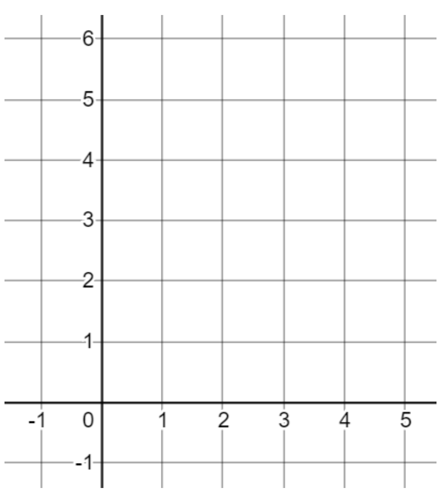


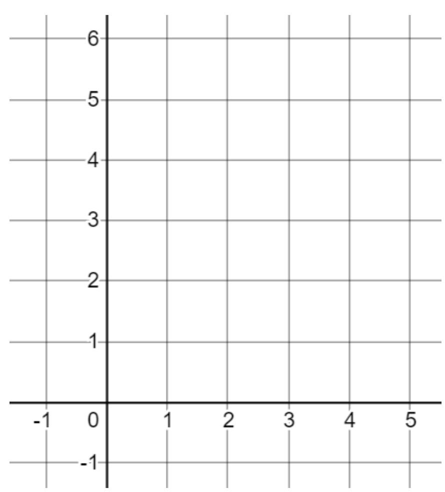
Practice Questions

3) Graph the following lines

a) $5x + 2y = 10$

b) $x + y = 3$





Review Unit – Linear Equations

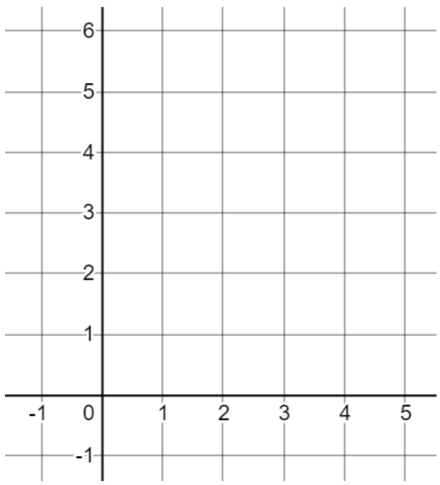
Sometimes we will have an equation of a line that only has one variable.

If the line only has an x (and not a y), put a dot on the x-axis at the number given and draw a vertical line.

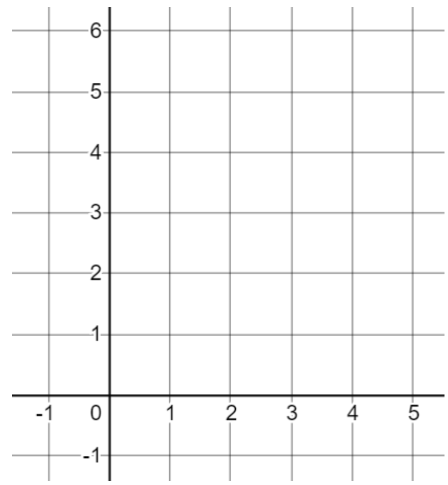
If the line only has a y (and not an x), put a dot on the y-axis at the number given and draw a horizontal line.

Ex: Graph the following lines.

a) $x = 2$



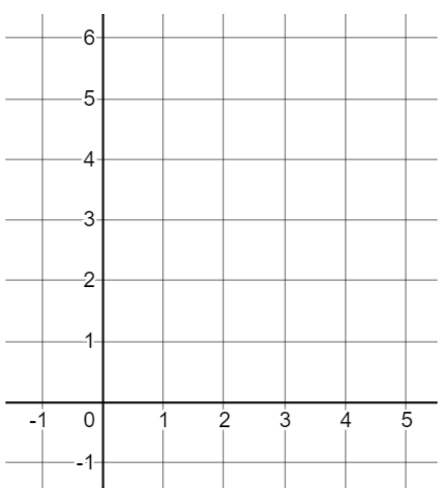
b) $y = 3$



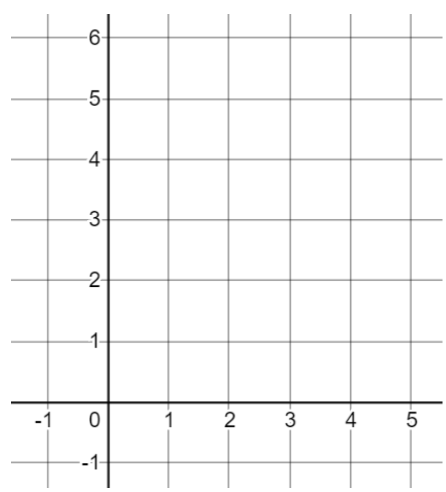
Practice Questions

4) Graph the following lines

a) $y = 5$



b) $x = 4$



Practice Questions

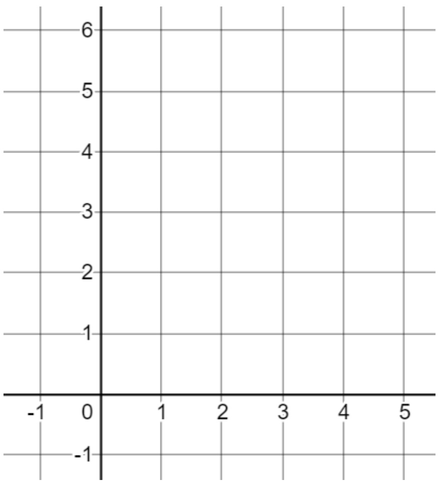
5) Find the slope and y-intercept of the line:

$$y = \frac{2}{3}x - 10$$

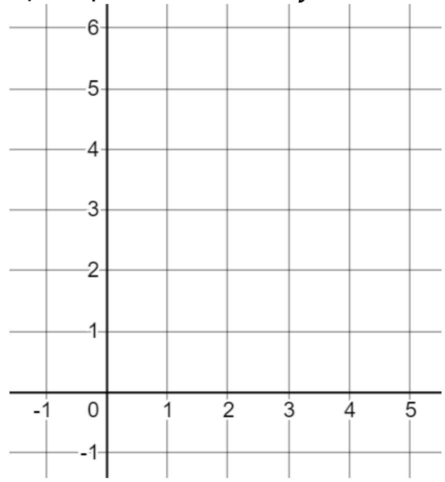
6) Find the slope and y-intercept of the line:

$3x - 7y = 35$

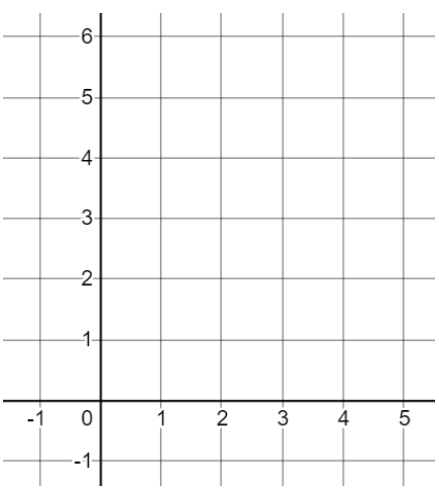
7) Graph the line: $y = 3x - 1$



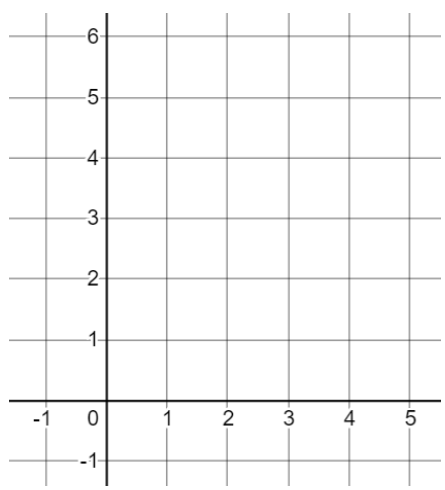
8) Graph the line: $3y + x = 15$



9) Graph the line: $x = 1$



10) Graph the line: $y = 4$



Review Unit – Linear Systems

R.3 LINEAR SYSTEMS

A linear system is when we have two lines.

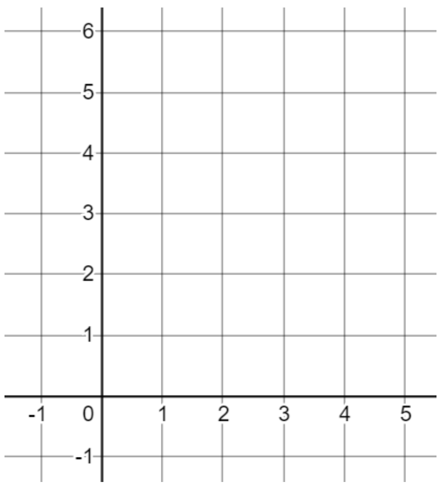
The **solution** is the point (x, y) where the two lines cross each other. We can find the solution to a system of equations graphically or algebraically (using elimination, comparison, or substitution).

Using a graph to solve:

The find a solution, graph both lines. The solution is the point where the lines cross, written as an ordered pair (x, y). Remember, you may need to re-arrange the equation before you can graph it.

Ex: Find the solution to the linear system.

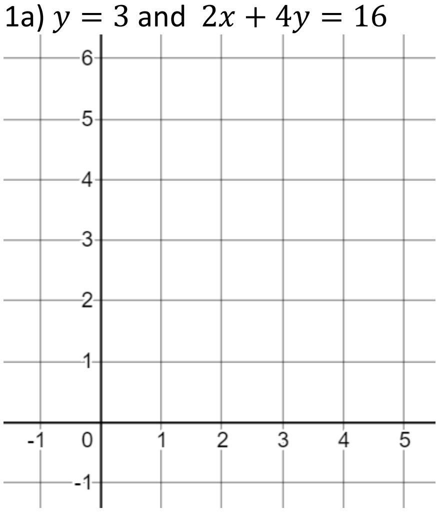
$y = 2x + 1$ and $x + y = 4$



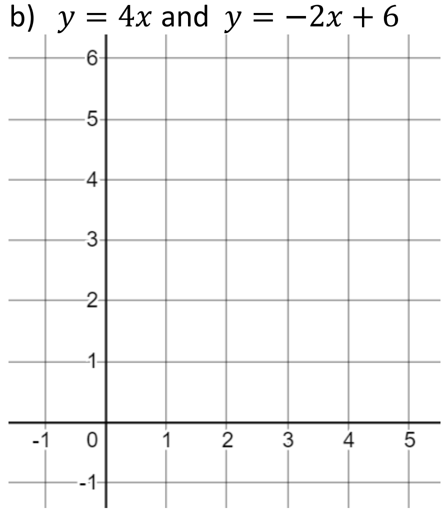
Practice Questions

1) Find the solution to the linear systems.

1a) $y = 3$ and $2x + 4y = 16$



b) $y = 4x$ and $y = -2x + 6$



Review Unit – Linear Systems

The graphing method is not an accurate way to solve linear systems. For example, it is difficult to tell the different between $(2.3, 4.6)$ and $(2.2, 4.7)$. Therefore, we will use algebra to solve linear systems.

There are 3 methods we can use: elimination, comparison, and substitution.

Using the elimination method to solve:

- Both lines must be in the form $Ax + By = C$.
- Multiply the entire first equation by the coefficient of x in the second equation.
- Multiply the entire second equation by the coefficient of x in the first equation, but change the sign.
- Add the two equations.
- Solve for the remaining variable.
- Use the solution in either equation to solve for the other variable.
- Write the solution (x, y).

Ex: Find the solution to the linear systems.

a) $2x + 5y = 16$ and $3x - 4y = 1$

b) $4x - 5y = 10$ and $y = -\frac{5}{3}x + 35$

Practice Questions

2) Find the solution to the linear systems.

a) $2x + 5y = -4$ and $3x - 2y = 13$

b) $3x + 4y = -6$ and $y = -2x + 1$

Review Unit – Linear Systems

Using the comparison method to solve:

- Both lines must be in the form $y = ax + b$.
- Take the $ax + b$ pieces from each equation and set them equal to each other $ax + b = ax + b$.
- Solve for x.
- Use either equation (and the value of x you just found) to solve for y.
- Write the solution (x, y).

Ex: Find the solution to the linear systems.

a) $y = 2x + 1$ and $y = -1.5x + 4.5$

b) $y = -2x - 6$ and $5x + y = -3$

Practice Questions

3) Find the solution to the linear systems.

a) $y = 2x + 5$ and $y = -4x + 11$

b) $y = 0.5x + 2$ and $y - 2x = -1$

Review Unit – Linear Systems

Using the substitution method to solve:

- This method works best if we already know the value of x or y.
- Use the equation that has both variables and replace the known variable.
- Solve for the missing variable.
- Write the solution as (x, y).

Ex: Find the solution to the linear systems.

a) $x = 2$ and $y = 3x + 8$

b) $y = 3$ and $3x + 4y = 20$

Practice Questions

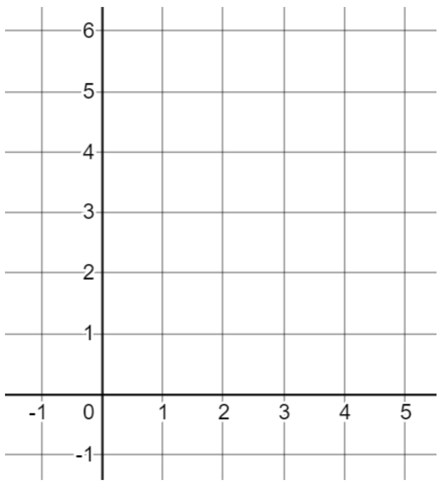
4) Find the solution to the linear systems.

a) $y = 5$ and $y = 2x - 15$

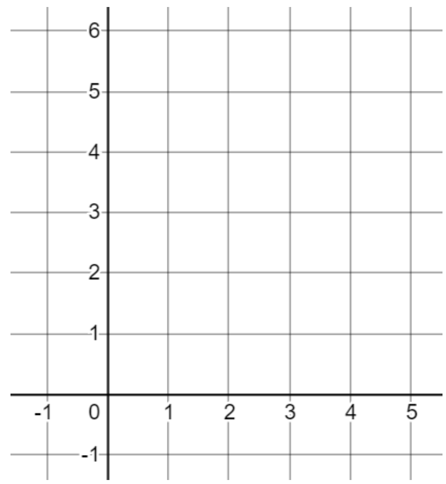
b) $x = 4$ and $3x + 2y = 20$

Practice Questions

6) Solve the system using graphing:
$y = 2x - 1$ and $y = x + 1$



7) Solve the system using graphing:
$y = -3x + 4$ and $y + 2 = 3x$



8) Solve the system using elimination:
$8x - 6y = -20$ and $-16x + 7y = 30$

9) Solve the system using elimination:
$-4y - 11x = 36$ and $20 = -10x - 10y$

Review Unit – Linear Systems

10) Solve the system using comparison:
$y = x - 13$ and $y = -2x + 5$

11) Solve the system using comparison:
$y = -4x + 2$ and $x - y = 3$

12) Solve the system using substitution:
$y = -5$ and $5x + 4y = -20$

13) Solve the system using substitution:
$x = 3$ and $4x - y = 20$

Optimization Unit – Definition and Steps

1.1 OPTIMIZATION DEFINITION AND STEPS

Mathematics can help us figure out optimal solutions. Linear optimization helps us determine how to achieve maximum profit or minimum cost (for example).

Optimization is a long process, but we'll take it one step at a time.

Steps for Optimization:

1. Define variables ("Let x be… Let y be…")
2. Turn words into inequalities
3. Rearrange into $y = ax + b$ form
4. Graph and shade all inequalities
5. Identify polygon of constraints
6. Find (x, y) of each vertex (corner) of the polygon of constraints
7. Write optimizing function
8. Use optimizing function at each vertex
9. Choose maximum or minimum (depending on the question)
10. Write concluding sentence

1.2 OPTIMIZATION STEP 1

Step 1

The first step of optimization is defining our variable. To do this, we will read through the question and figure out the two things we're talking about.

Once we know the two things we care about, we will write two statements:

- "Let x be…"
- "Let y be…"

Note: I typically use x and y as the variables. This will make it easier when we start graphing.
I also usually let x be the first variable mentioned in the question and y be the second variable mentioned, but the final answer won't change if you swap them.

Ex: Define the variables for the following statement. There are red and blue marbles in a jar.

Practice Questions

1) Define the variables in each of the scenarios below.

a) A pencil case contains pens and pencils.

b) Students are holding a car wash where they wash cars and trucks.

c) A garden grows red tulips and white tulips.

Optimization Unit – Step 2

1.3 OPTIMIZATION STEP 2

Now that we've defined our variables, that will be the first thing we do in every optimization question. The next step is to turn statements into inequalities.

First, we'll look at turning statements into equalities.

Option 1: We are given a total amount. We will write the equation: $x + y = total$

Ex: Define the variables and turn the statement into an equation.
A bag contains red and blue marbles. There are a total of 15 marbles in the bag.

Practice Questions

1) Write an equation for each of the following scenarios

a) A garden grows roses and tulips. There are a total of 300 plants in the garden.

b) Students are holding a bake sale. They sell cookies and cupcakes. A total of 250 treats are sold.

c) John collects stuffed dinosaurs and stuffed bears. He has a total of 25 stuffed animals in his collection.

Optimization Unit – Step 2

Option 2: We are given a comparison statement. When we write the equation, x will be on one side and y on the other.

Hint: When reading comparison statements, define variables. See which variable comes first in the comparison statement. Write that variable on the left and the other variable on the right. Then figure out which thing you have more of. The multiplication (or addition) will go with the OTHER variable.

Ex: Define the variables and turn the statement into an equation.
A school is selling strawberry plants and tomato plants as a fundraiser. They sell twice as many strawberry plants as tomato plans.

Ex: Define the variables and turn the statement into an equation.
Students are holding a fundraiser washing cars and trucks. They wash 10 more trucks than cars.

Practice Questions

2) Write an equation for each of the following scenarios

a) A company sells road bikes and mountain bikes. They sell three times as many road bikes as mountain bikes.

b) Dr. James drinks tea and coffee. In any given week, she drinks 3 more teas than coffees.

c) A farm has goats and chickens. There are 5 times as many chickens as goats

Optimization Unit – Step 2

To change statements into inequalities, we will follow the same steps as changing statements to equalities, but instead of $=$ we will use $>, \geq, <, \text{or} \leq$.

Math Inequality Symbols and Words	
$<$	**Less Than** Is under Is fewer
$>$	**Greater Than** Is more than Is greater Exceeds
$\leq$	**Less Than or Equal To** Is at most Has a maximum of Is not greater than Does not exceed (go over) Is not more than
$\geq$	**Greater Than or Equal To** Is at least Is not less than Is not under Has a minimum value of

When we are converting words to inequalities, this chart can help us determine which inequality to use.

Hint: when turning statements into inequalities, make sure you write the variables in the same order as they appear in the statement.

Ex: Define the variables and change the statement into an inequality.
A family has more dogs than cats.

Ex: Define the variables and change the statement into an inequality.
A food truck sells hotdogs and hamburgers. Every day they sell at least 100 items.

Optimization Unit – Step 2

Ex: Define the variables and change the statement into an inequality.

Reese collects international and domestic stamps. Reese has no more than twice as many domestic stamps as international stamps.

Practice Questions

3)Write an inequality for each of the following scenarios

a) Students are selling bracelets and necklaces as a fundraiser. They sell at least 5 more bracelets than necklaces.

b) Sidney is making homemade soaps to sell at a local market. Sidney makes citrus scented soap and lavender scented soap. Sidney expects to sell fewer than 100 total soaps.

c) Ms. Stinger is selling large and small jars of honey. She can make no more than twice as many small jars as large jars.

Practice Problems

4) A manufacturing plant produces cars and truck. Define the variables and translate each of the following statements into equalities or inequalities.

a) A maximum of 200 vehicles are produced day.

b) The plant must produce at least 100 vehicles each day.

c) Fewer than twice as many cars as trucks are produced.

d) The plant produces at least 40 cars each day.

e) The number of trucks produced must not exceed 150.

Optimization Unit – Step 3

1.4 OPTIMIZATION STEP 3

After turning statements into inequalities we need to graph those inequalities. Before we can do that, we must rearrange them into $y = ax + b$ form so we can find the initial value and the slope.

However, we are going to keep the inequality symbol $(<, >, \leq, or\ \geq)$. We can rearrange inequalities just like we would rearrange equations with $=$, unless we multiply or divide by a negative number.

Ex: Define the variables, turn the statement into an equation, and rearrange into $y = ax + b$ form.
A bag contains red and blue marbles. There are at least 8 marbles in the bag.

Ex: Define the variables, turn the statement into an inequality, and rearrange into $y = ax + b$ form.
A bag contains red and blue marbles. There are no more than twice as many red marbles as blue marbles.

Practice Problems

1) Define the variables, turn the statement an inequality and, and rearrange into $y = ax + b$ form.

a) Students are holding a bake sale. They sell cookies and cupcakes. No more than 75 treats are sold.

b) John collects stuffed dinosaurs and stuffed bears. He has at least three times as many dinosaurs as bears.

c) John collects stuffed dinosaurs and stuffed bears. He has fewer than half as many bears as dinosaurs.

If an inequality only has one variable (either x or y, but not both), make sure the variable is on the left side of the inequality.

Ex: Define the variables, turn the statement into an inequality, and rearrange into $y = ax + b$ form.
In a school, students choose between art and music. At least 200 students sign up for music.

Practice Problems

2) A café is selling tea and coffee. Define the variables, translate each of the following statements into inequalities, and rearrange into $y = ax + b$ form.

a) The café sells no more than 700 drinks in a day.

b) Every day the café sells at least 350 coffees.

c) The café sells fewer than twice as many coffees as teas.

d) A maximum of 200 teas are sold each day.

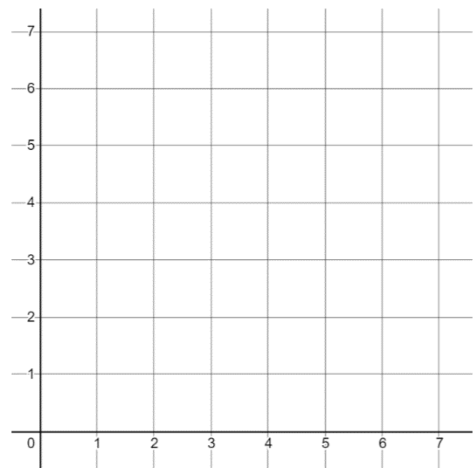
Optimization Unit – Step 4

1.5 OPTIMIZATION STEP 4

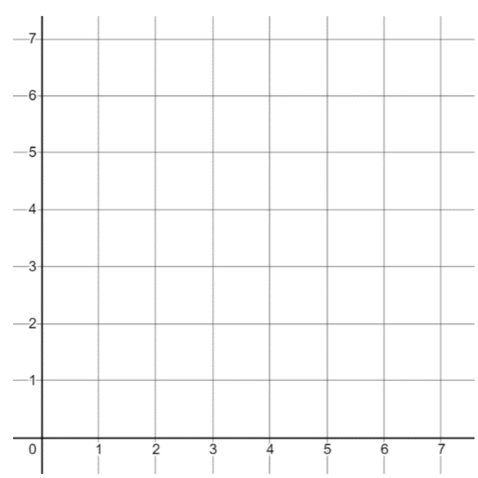
There are 3 steps to graphing an inequality (once it's been rearranged into $y = ax + b$ form).

1) Determine whether to use a solid line or a dotted line.
    a. Use a solid line if $\leq$ or $\geq$
    b. Use a dotted line if $<$ or $>$
2) Graph as usual (putting a dot on the y-axis at b and then using slope to find a second point).
3) Shade either above the line or below the line
    a. Shade above the line (draw arrows straight up) if $> or \geq$
    b. Shade below the line (draw arrows straight down) if $< or \leq$

Ex: Define the variables, turn the statement into an equation, rearrange into $y = ax + b$ form and graph.
A bag contains red and blue marbles. There are at least 6 marbles in the bag.



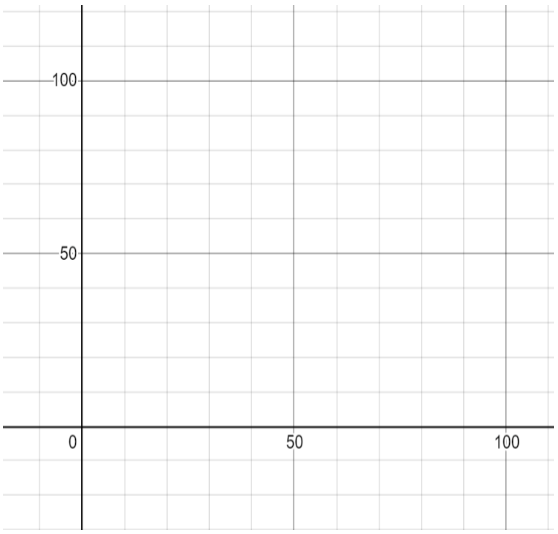
Ex: Define the variables, turn the statement into an equation, rearrange into $y = ax + b$ form and graph.
A bag contains red and blue marbles. There are more than twice as many red marbles as blue marbles.



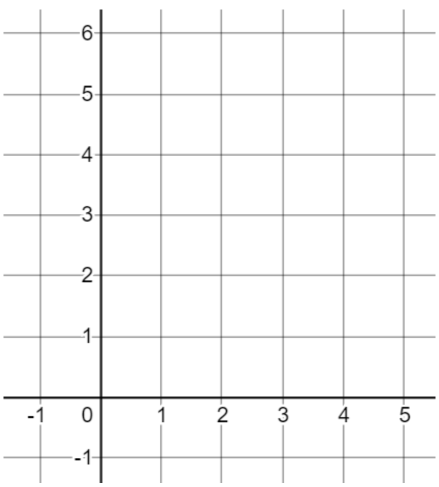
Practice Problems

1) Define the variables, turn the statement into an equation, rearrange into $y = ax + b$ form and graph.

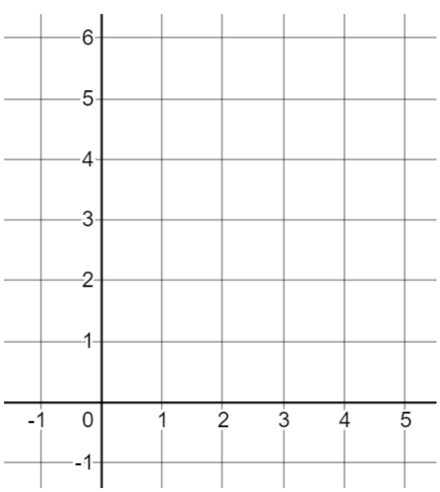
a) Students are holding a bake sale. They sell cookies and cupcakes. More than 75 treats are sold.



b) John collects stuffed dinosaurs and stuffed bears. He has at least three times as many dinosaurs as bears.



c) John collects stuffed dinosaurs and stuffed bears. He fewer than half as many bears as dinosaurs.

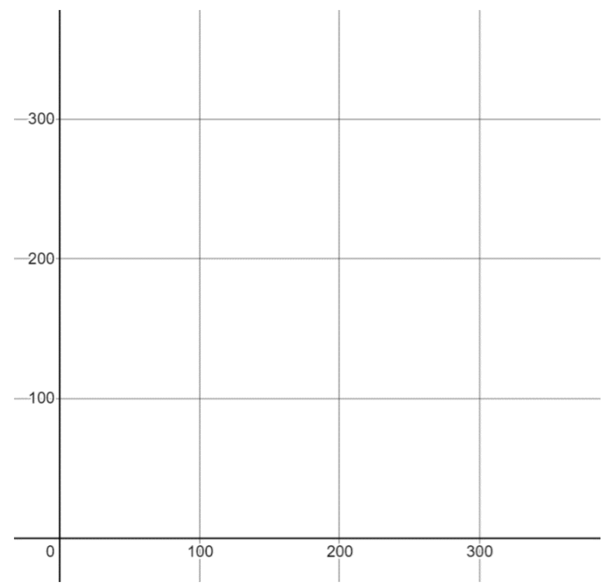


Optimization Unit – Step 4

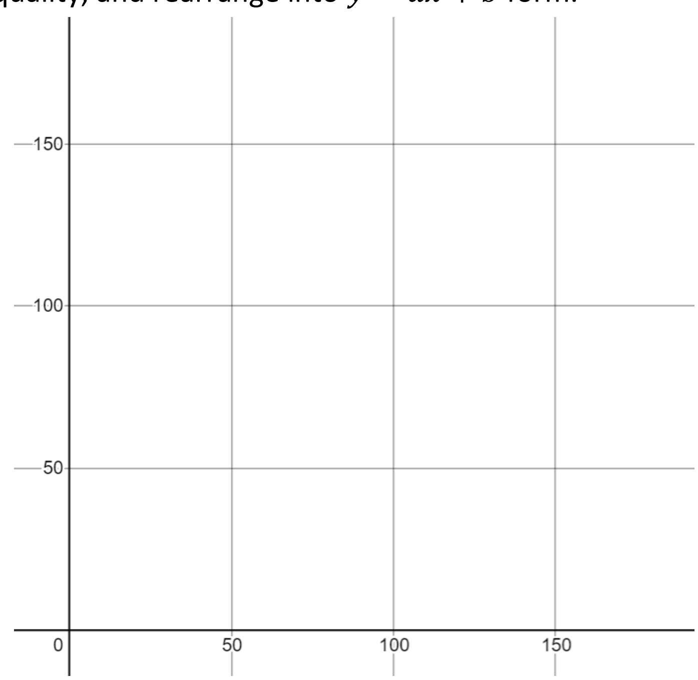
If you only have one variable, isolate the variable on the left side and graph and shade as usual, except:

- If $x >$ or $x \geq$ shade to the right of the line (or draw arrows to the right)
- If $x <$ or $x \leq$ shade to the left of the line (or draw arrows to the left)

Ex: Define the variables, turn the statement into an inequality, and rearrange into $y = ax + b$ form.
In a school, students choose between art and music. At least 200 students sign up for music.



Ex: Define the variables, turn the statement into an inequality, and rearrange into $y = ax + b$ form.
In a school, students choose between art and music.
Fewer than 100 students sign up for art.

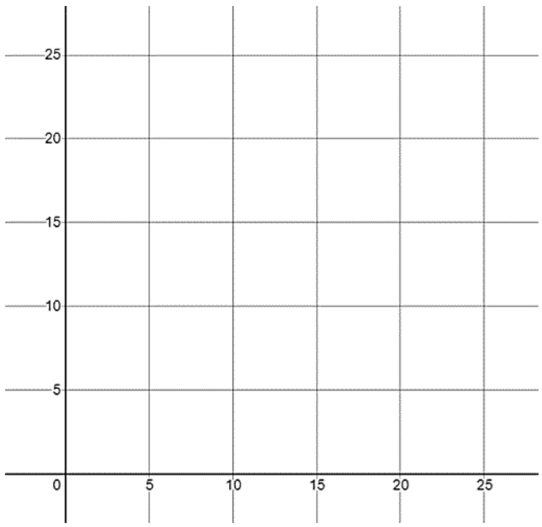


Practice Problems

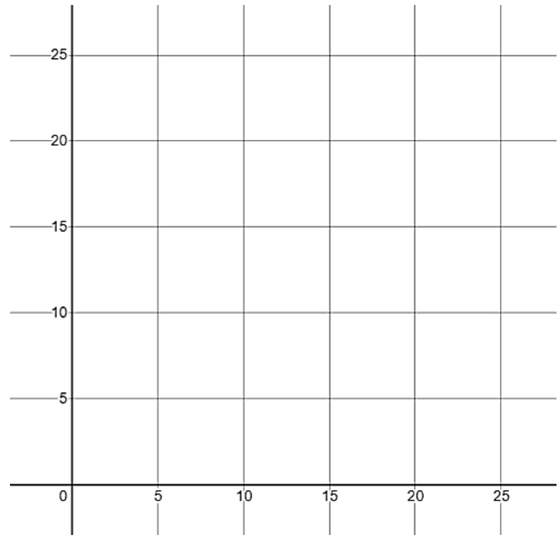
2) Define the variables, translate each of the following statements into inequalities, and rearrange into $y = ax + b$ form and graph for the scenario below.

Lisa, a Grade 11 student, is fundraising for Prom. She sells strawberry baskets and flower baskets.

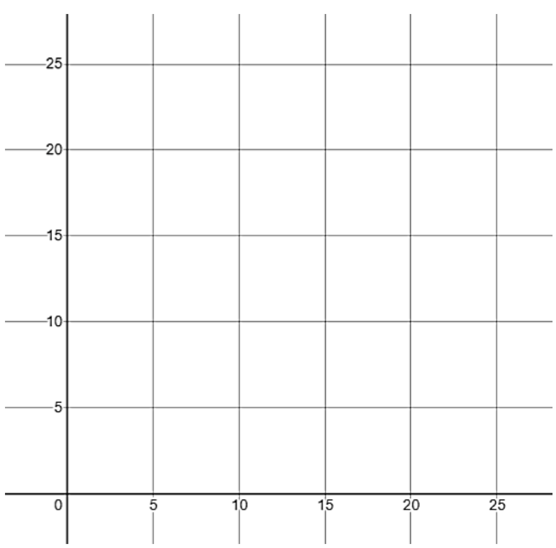
a) A maximum of 24 baskets can be sold.



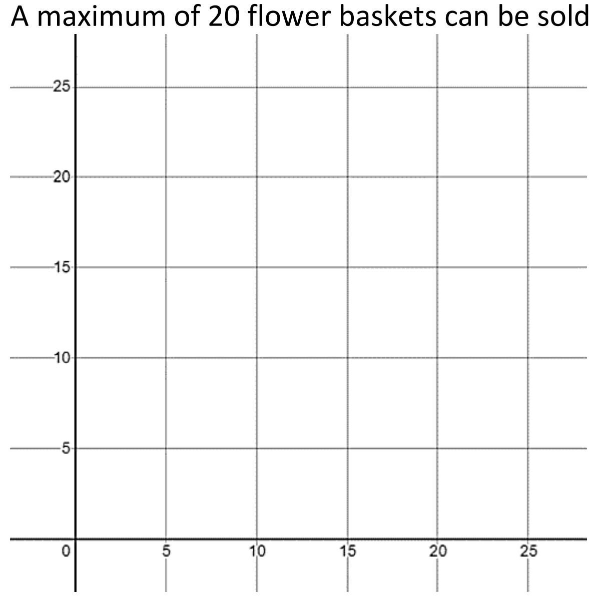
b) A minimum of 5 strawberry baskets must be sold.



c) The number of flower baskets must be at least triple the number of strawberry baskets sold.



d) A maximum of 20 flower baskets can be sold.



1.6 OPTIMIZATION STEP 5

When we have more than one linear inequality, we can graph them all together. This is a system of linear inequalities.

The solution to a system of inequalities is where all the shading overlaps.

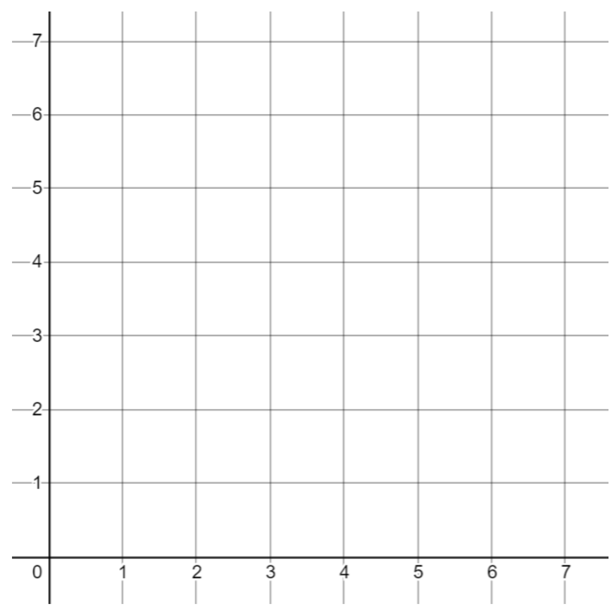
A **polygon of constraints** is when the overlapping shading is bound on all sides by a line.

There are 3 steps to graphing a system of inequalities:

4) Graph and shade all lines (instead of shading, it can help to use arrows).
5) Determine the polygon of constraints by determining the area where all shading overlaps.
6) Shade the polygon of constraints.

Ex: Determine the polygon of constraints given the following scenario:
A bag contains red and blue marbles.

- There are no more than 4 blue marbles.
- There is at least 1 red marble.
- There is a maximum of 6 marbles.

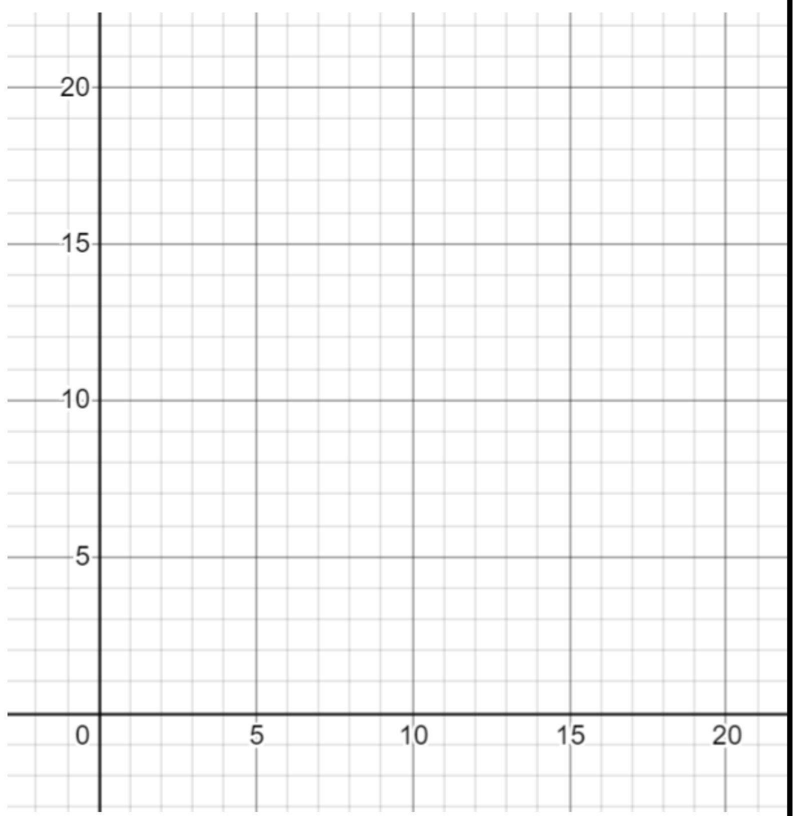


Practice Problems

1) Determine the polygon of constraints given the following scenario.

John loves his stuffed animals. His collection has stuffed bears and stuffed dinosaurs.

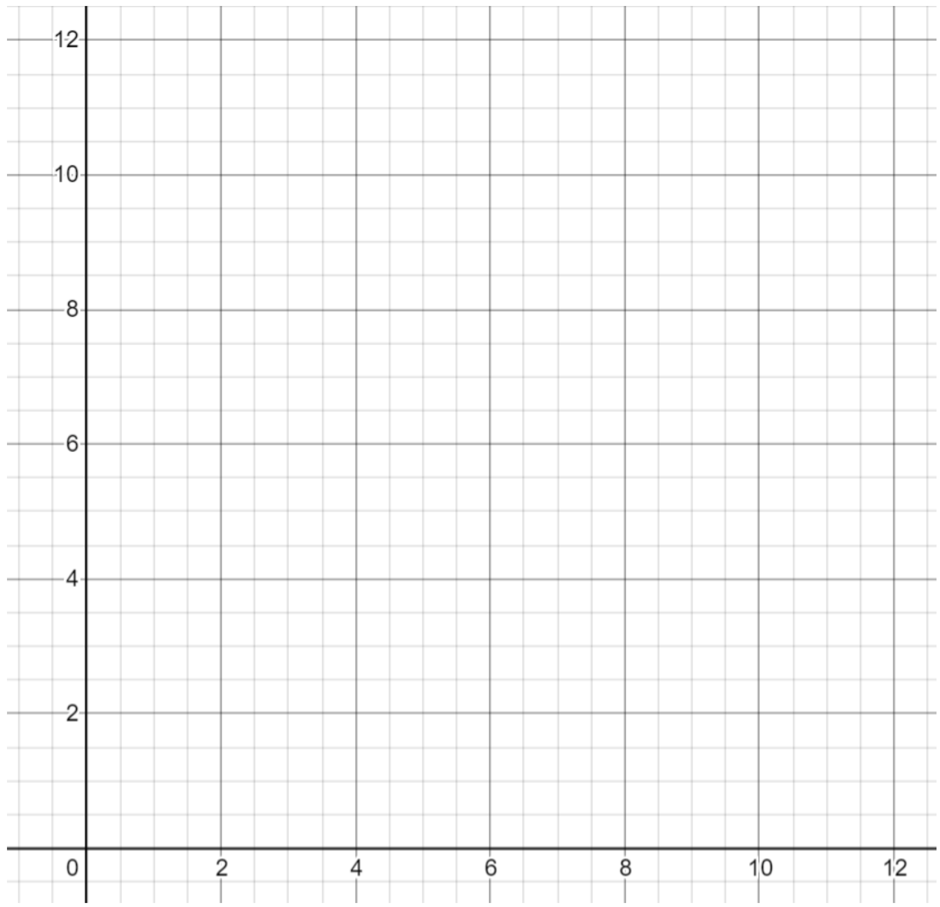
- John has a minimum of 10 stuffed animals.
- John has a maximum of 20 stuffed animals.
- John has less than or equal to 6 bears.
- John has at least 5 more dinosaurs than bears.



2) Determine the polygon of constraints given the following scenario.

Sandra and Jane run a pet spa. They offer two services: dog grooming and cat grooming.

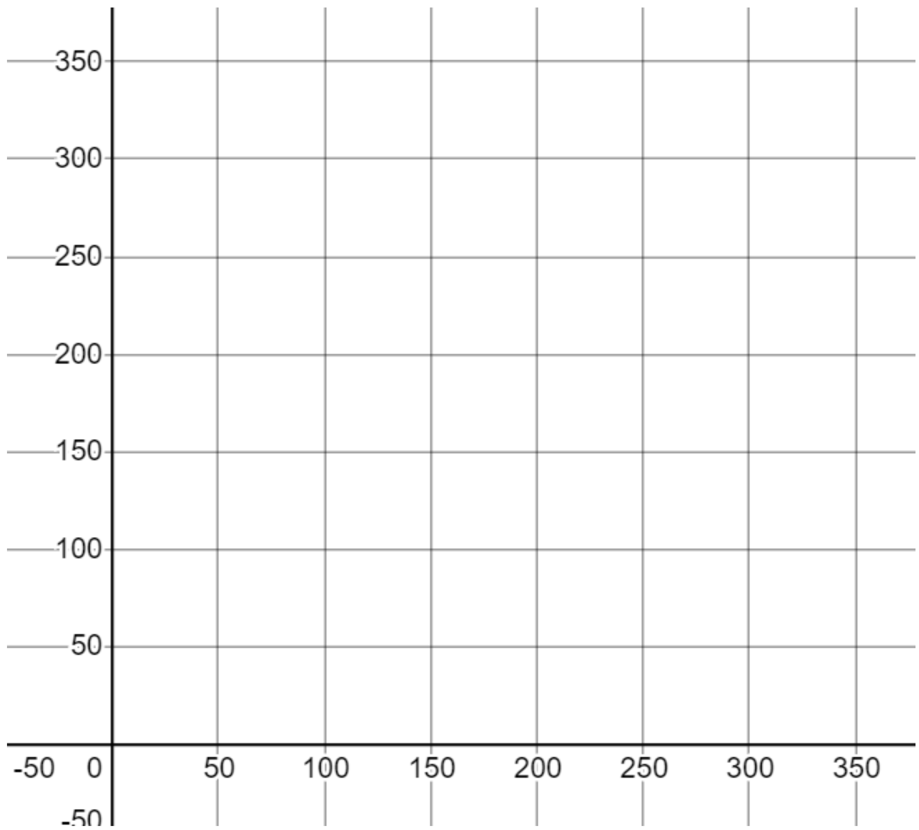
- They will bathe a maximum of 12 cats per day.
- They can bathe no more than 8 dogs per day.
- They will bathe a maximum of 3 times as many cats as dogs.



3) Determine the polygon of constraints given the following scenario.

To raise funds, members of an association organize a concert. They want to reserve some seats for donors and the other seats will be for general admission.

- The theater contains a maximum of 300 seats.
- There must be at least 3 times as many seats for general admission as there are seats reserved for donors.
- There must be at least 50 seats for donors.
- There is a maximum of 250 seats for general admission.



Optimization Unit – Step 6

1.7 OPTIMIZATION STEP 6

Now that we have the polygon of constraints, we need to find the vertices. That is, we need to find the (x, y) of each vertex (corner) of the polygon of constraints.

There are 3 steps to finding the vertices:

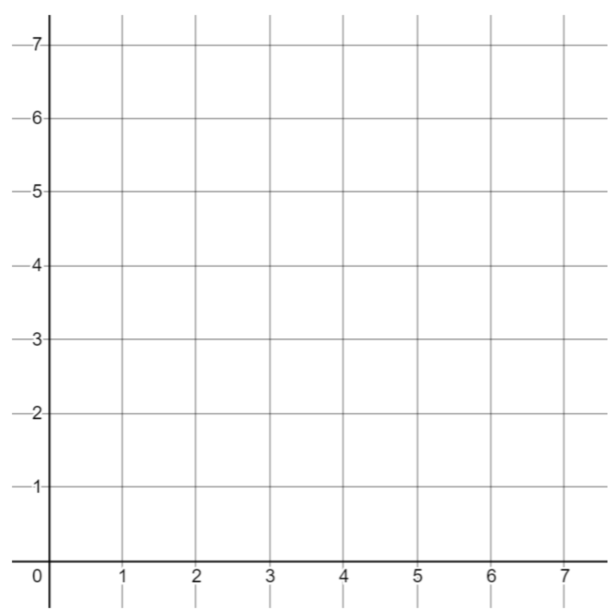
1) Label each vertex of the polygon of constraints (we usually use A, B, C, etc.).
2) Identify the two lines that go through each vertex and change inequalities to $=$
3) Solve the system

If the lines that intersect to form the vertex are:	Then use:
2 equations each including both variables	Comparison
1 equation with both variables and 1 equation with one variable	Substitution
2 equations each including one variable	No calculations necessary

Ex: Determine the vertices of the polygon of constraints given the following scenario:

A bag contains red and blue marbles.

- The bag contains a maximum of 6 marbles.
- The bag contains a minimum of 2 marbles.
- There are at least as many red marbles as blue marbles.
- The bag contains at most 5 red marbles.

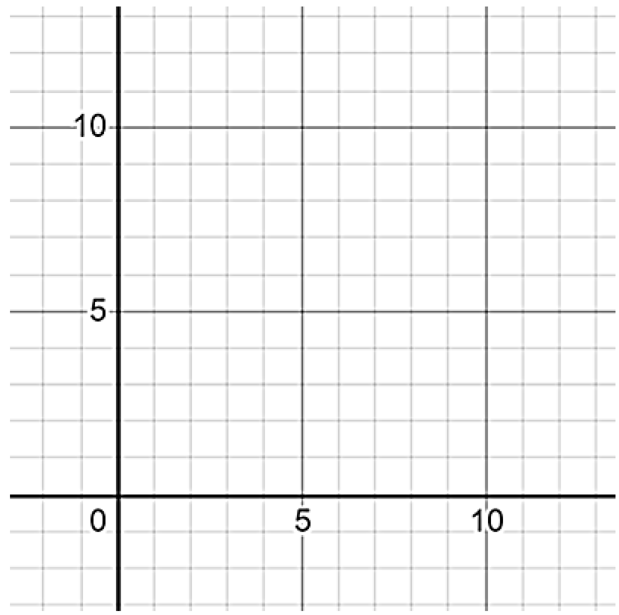


Practice Question

1) Determine the vertices of the polygon of constraints given the following scenario:

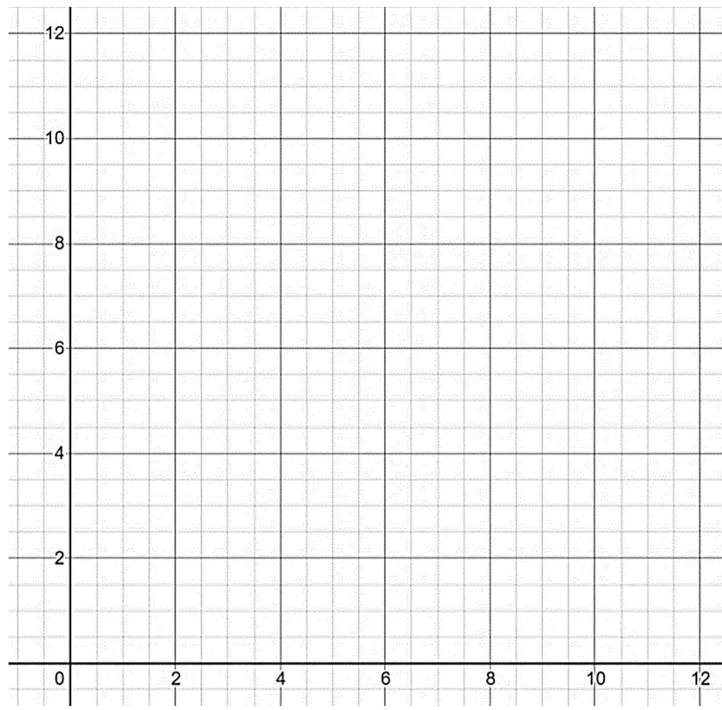
John loves his stuffed animals. His collection has stuffed bears and stuffed dinosaurs.

- John has a maximum of 10 stuffed animals.
- John has less than or equal to 4 bears.
- John has at least 1 more dinosaur than bear.



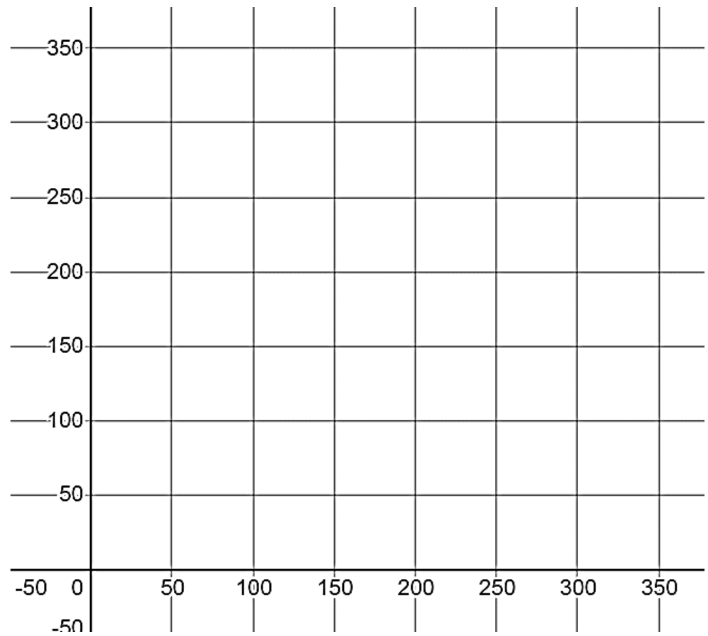
2) Sandra and Jane run a pet spa. They offer two services: dog grooming and cat grooming.

- They will bathe a maximum of 12 cats per day.
- They can bathe no more than 8 dogs per day.
- They will bathe a maximum of 3 times as many cats as dogs.



3)To raise funds, members of an association organize a concert. They want to reserve some seats for donors and the other seats will be for general admission.

- The theater contains a maximum of 300 seats.
- There must be at least 3 times as many seats for general admission as there are seats reserved for donors.
- There must be at least 50 seats for donors.
- There is a maximum of 250 seats for general admission.



1.8 STEPS 7-10

The **target objective** is the search for the optimal solution. It is either the search for the highest value (maximum) or lowest value (minimum). The optimal value is obtained by using the **optimizing function**.

In each optimization question, you will be given a statement about money (cost, profit, etc.). This will become the optimizing function.

When you substitute each vertex into the optimizing function (one at a time), one vertex will give you the maximum and one vertex will give you the minimum.

- **Step 7** is to identify the target objective and write the optimizing function. Go back to the question and pick out the sentence that has to do with money. Turn this into an equation.
  **Be Careful! You won't have to use a statement about money until this step, so ignore it when you are creating your graph and finding your vertices.*

- **Step 8** is to use the optimizing function at each vertex. Take each vertex, one at a time, and replace the x and y in the optimizing function with the (x, y) coordinates of the vertex.

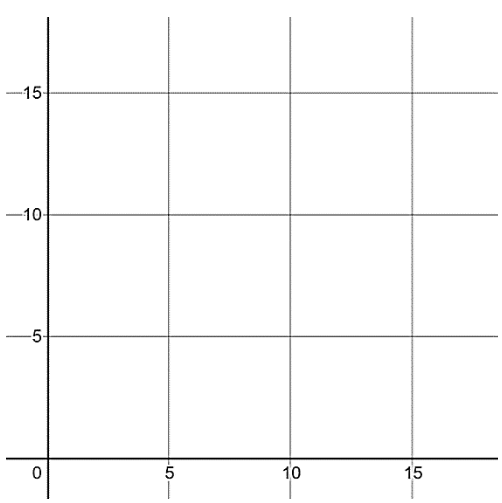
- **Step 9** is to choose the maximum or minimum depending on the vertex. One of the vertices will give you the maximum. One of the vertices will give you the minimum.

- **Step 10** is to write a concluding statement answering the question. The question may ask of the maximum, minimum, or the number of items that give you a maximum or minimum.

Ex: Determine the vertices of the polygon of constraints given the following scenario:
A bag contains red and blue marbles.

- The bag contains a maximum of 12 marbles.
- The bag contains a minimum of 6 marbles.
- There are no more than twice as many red marbles as blue marbles.
- The bag contains at least 3 red marbles.

If each red marble is worth $1.50 and each blue marble is worth $3.00, what is the maximum value of the collection of marbles in the bag? How many of each marble do you need to have the maximum value?



Optimization Unit – Steps 7-10

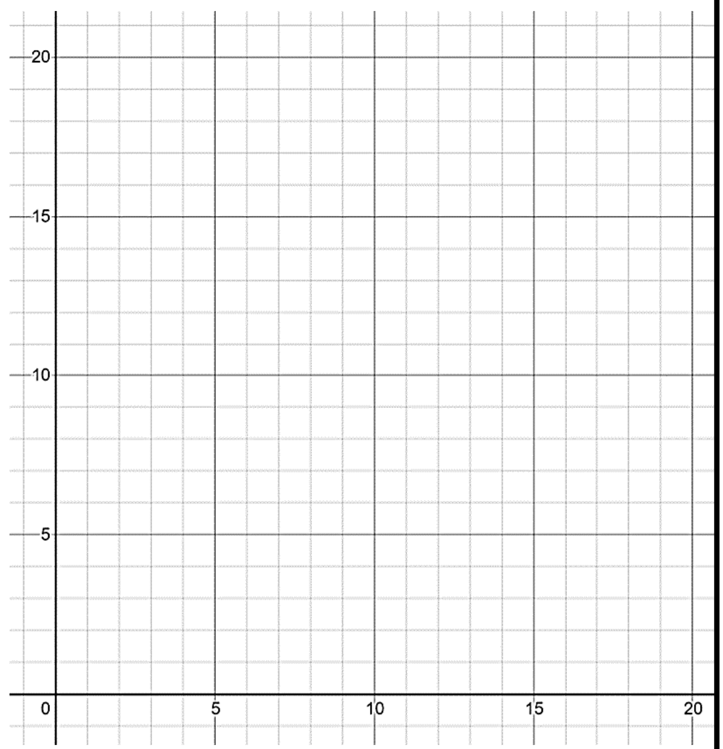
Practice Questions

1) Determine the vertices of the polygon of constraints, the optimizing function, and the optimal solution.

John loves his stuffed animals. His collection has stuffed bears and stuffed dinosaurs.

- John has a maximum of 20 stuffed animals.
- John knows he has at least 10 stuffed animals.
- John has less than or equal to 6 bears.
- John has at least 5 more dinosaurs than bears.

Given that each bear is worth \$10 and each dinosaur is worth \$5, what is the minimum value of John's collection and how many of each animal would John have if his collection was worth the minimum value?

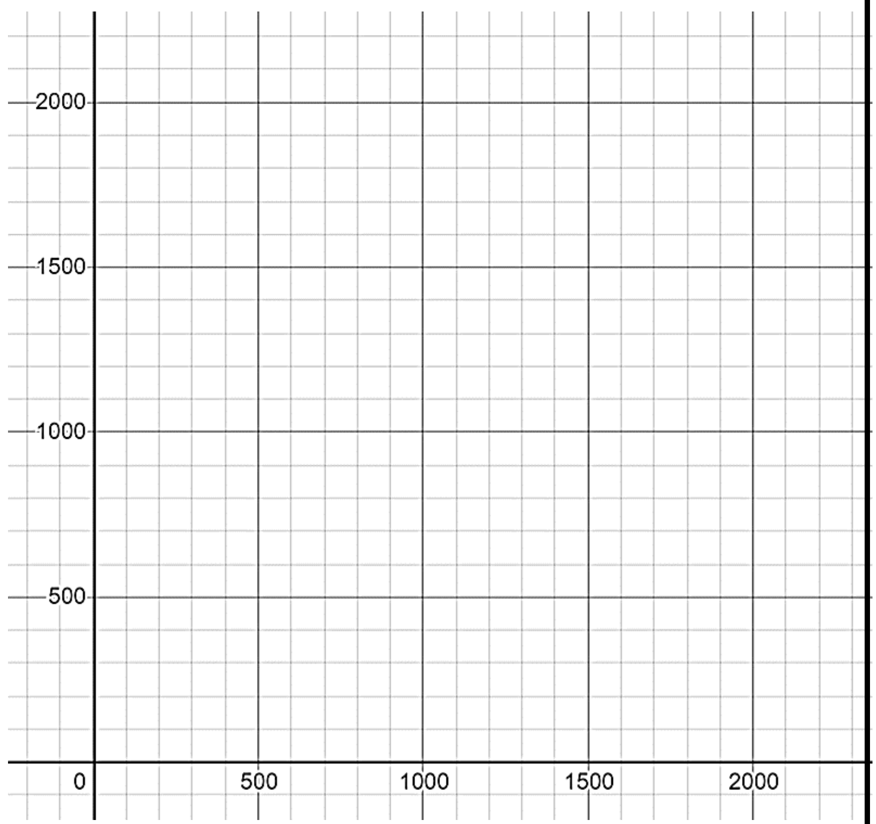


2) Determine the vertices of the polygon of constraints, the optimizing function, and the optimal solution.

A car manufacturer builds compact cars and minivans and wants to maximize its weekly profit. The profit generated from each car is $4000 and the profit generated from each minivan is $10 000.

- The manufacturer's weekly production capacity is 2100 at most.
- The manufacturer must build at least 1000 compact cars weekly.
- The manufacturer must build at least 200 minivans weekly.
- The number of compact cars built each week must be at least twice as many as the number of minivans built each week.

What is the maximum profit the manufacturer can earn weekly?



3) Determine the vertices of the polygon of constraints, the optimizing function, and the optimal solution.

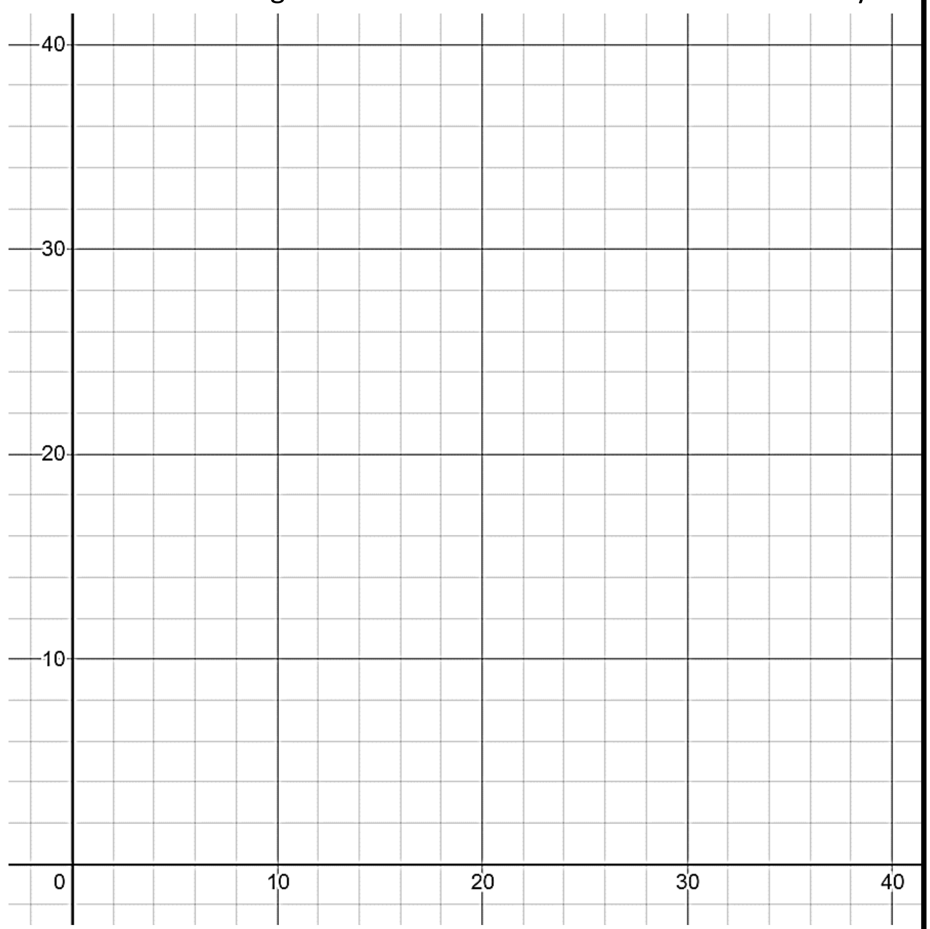
In order to treat a patient, a doctor decides to administer a treatment combining two medications, A and B. The side effects associated with these medications force the doctor to respect the following constraints:

- The dosage of medication A must be at least 5 mg.
- The dosage of medication A cannot exceed 15 mg.
- The dosage of medication B must be at least 8 mg.
- The dosage of medication B cannot exceed 25 mg.
- The total dosage of medication cannot exceed 35 mg.

The doctor wants the medications to be as effective as possible. Efficacy can be determined by using the following optimizing function: $Efficacy = 0.0305x + 0.025y$ where:

- x is the amount of medication A
- y is the amount of medication B

What combination of medications should the doctor give in order to achieve the maximum efficacy?



1.9 OPTIMIZATION COMPLICATIONS: DECIMALS, DOTTED LINES, and TIES

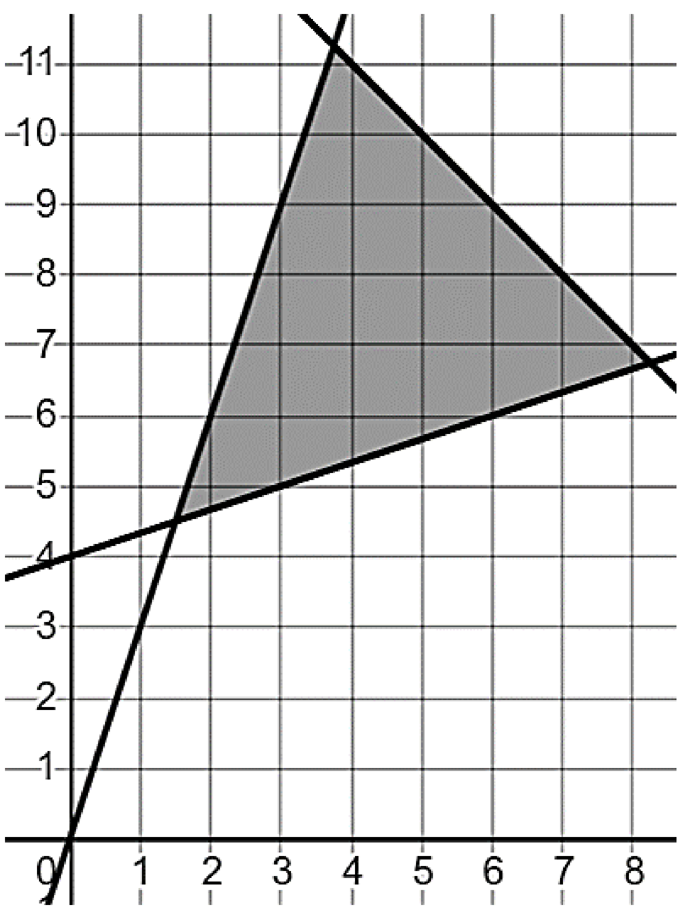
There are three potential complications to optimization: decimals, dotted lines, and ties.

Decimals

In optimization, the vertices always need to be whole numbers (not decimals). This is because we are talking about things that can only be produced or sold (for example) in whole units.

When we solve for the vertices, we always need to find a whole number (not a decimal). Sometimes we will get a decimal. When this happens, find a point near the vertex where x and y are both whole numbers either in the shaded area or on a solid line. Use that point instead of the decimals.

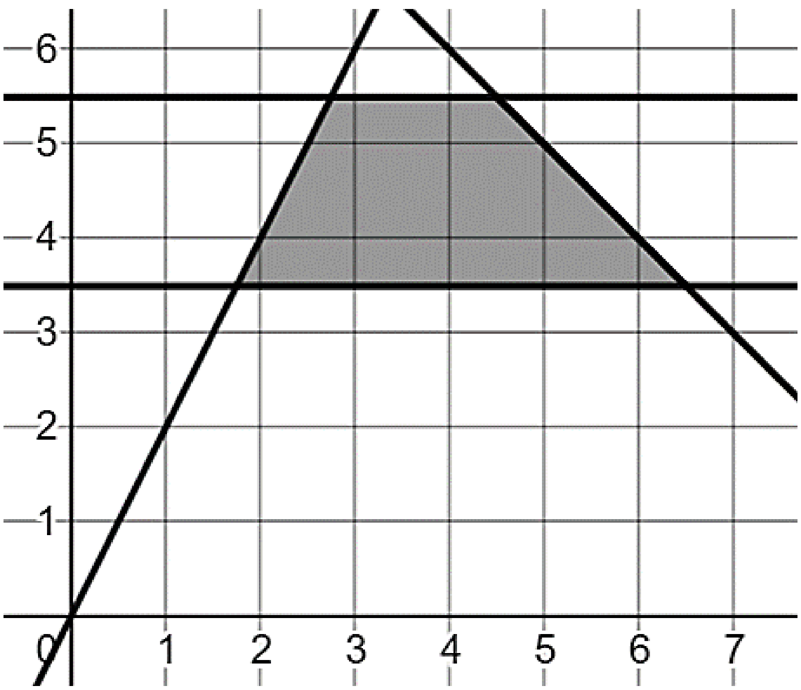
A school is holding a car wash and charges $5 to wash a car (x) and $8 to wash a truck (y). The students can only wash a limited number of vehicles per hour. The polygon of constraints is shown in the graph below. What is the maximum value the students could earn per hour?



Practice Question

1) Reese wants to maximize the value of his marble collection. Reese collects red and blue marbles, but he has limited space and money to purchase marbles. The possible combinations of red marbles (x) and blue marbles (y) are shown in the polygon of constraints below.

Given that red marbles are worth \$2 and blue marbles are worth \$1.50, what is the maximum value of Reese's collection?



Optimization Unit – Complications: Decimals, Dotted Lines, and Ties

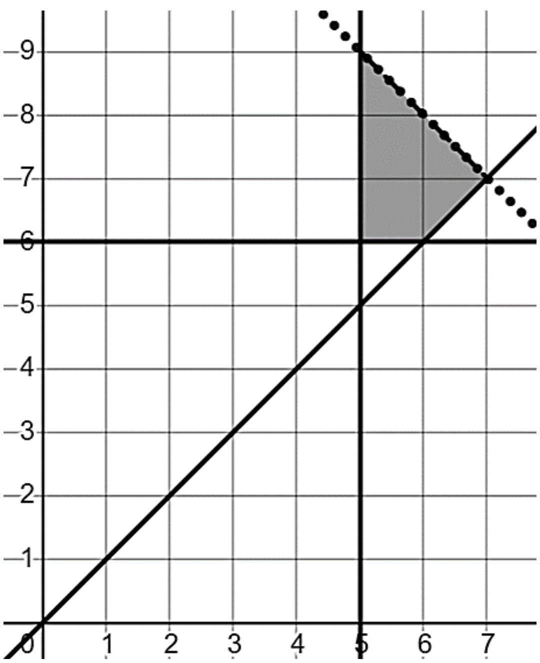
Dotted Lines

Dotted lines (which means the inequality has a $>$ or $<$ instead of a $\geq$ or $\leq$) presents another complication. If a vertex is on a dotted line, it cannot be the solution. This means the vertex itself is not included in the solution set, so cannot be the answer. However, a point close to the vertex could be the answer.

We will first solve this as if the line was not dotted. If the optimal solution is on a vertex with a dotted line, then we find a point close to that vertex and check to see if it's also an optimal solution. If not, we select a different vertex.

Ex: A coffee shop sells tea and coffee. The number of teas (x) and coffees (y) sold each hour is given by the polygon of constraints below.

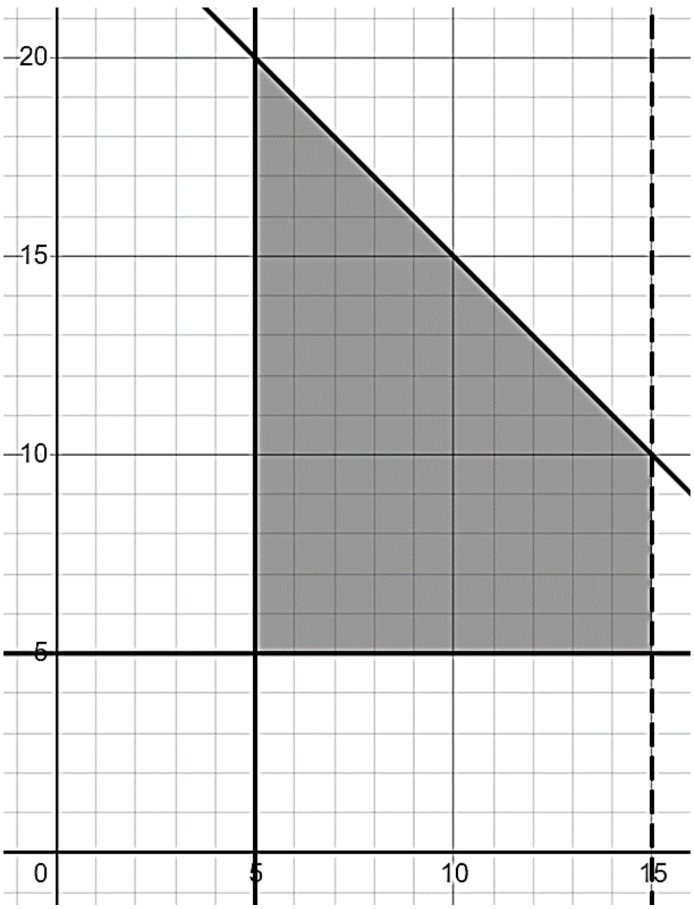
If the coffee shop earns a profit of \$0.75 for each tea sold and \$1.00 for each coffee sold. How many teas and coffees must the shop sell each hour in order to earn the maximum profit?



Practice Questions

2) A pizza shop sells whole pizzas (x) and slices of pizza (y). The possible combinations of whole pizzas and slices of pizza sold per hour is given in the polygon of constraints below.

Given that the shop earns a profit of $4 per whole pizza and $1.50 per slice of pizza, how many slices should the store per hour in order to earn the maximum profit?



Optimization Unit – Complications: Decimals, Dotted Lines, and Ties

Ties

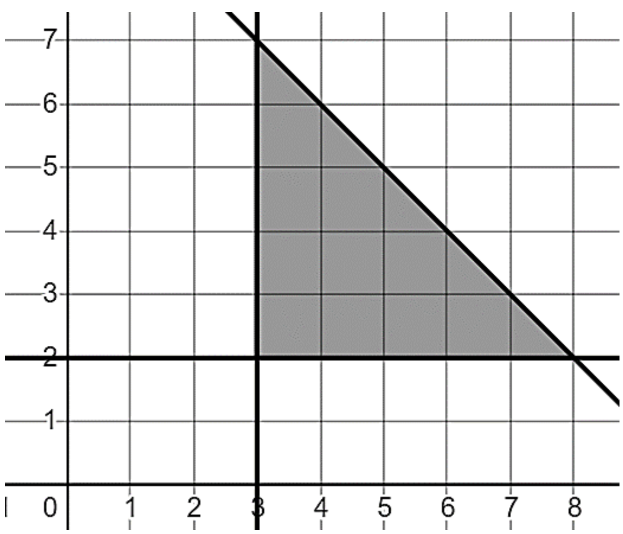
The final complication present in some optimization questions is when two vertices tie for a maximum or minimum after using the optimizing function.

When two vertices tie for the optimal value, they are both solutions to the scenario. In addition, all the points on the line connecting the two tied vertices are also optimal solutions.

Ex: Students at Philemon Wright are selling scented candles as a fundraiser for their annual trip. They are selling lavender scented candles (x) and vanilla scented candles (y). The possible combinations of lavender and vanilla candles sold each day are presented in the polygon of constraints below.

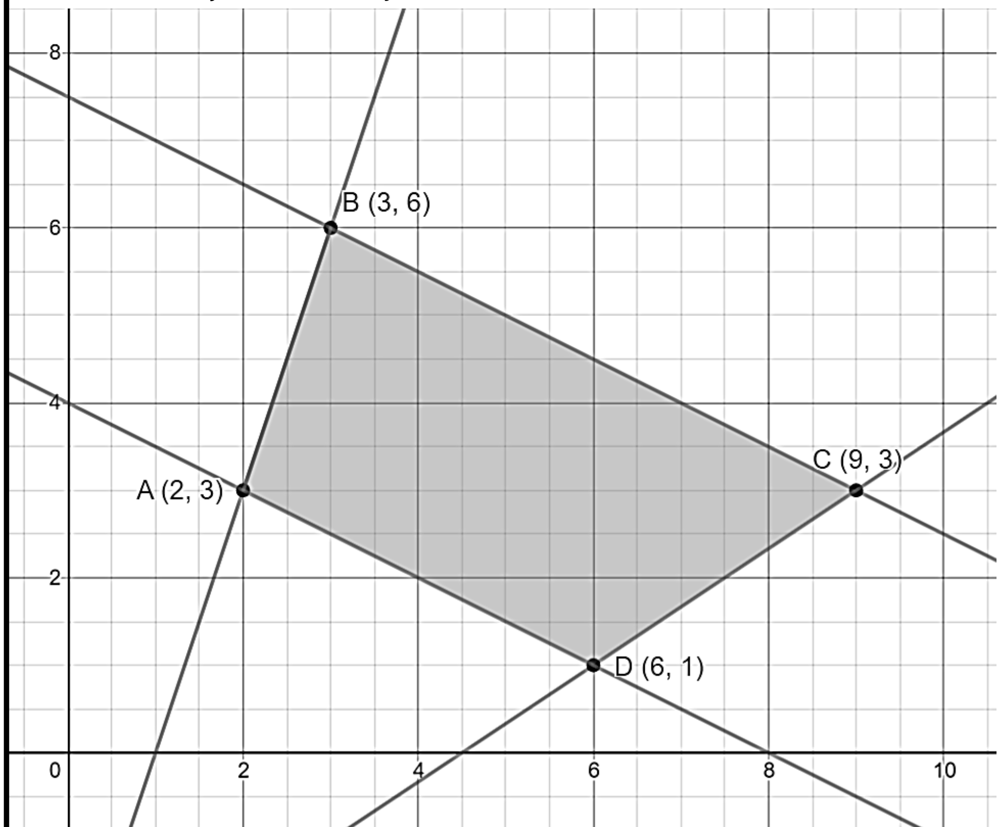
The profit for the lavender scented candles is $3 and the profit for the vanilla scented candles is also $3.

What is the maximum profit the students make each day and what are all the possible combinations of scents to maximize the value?

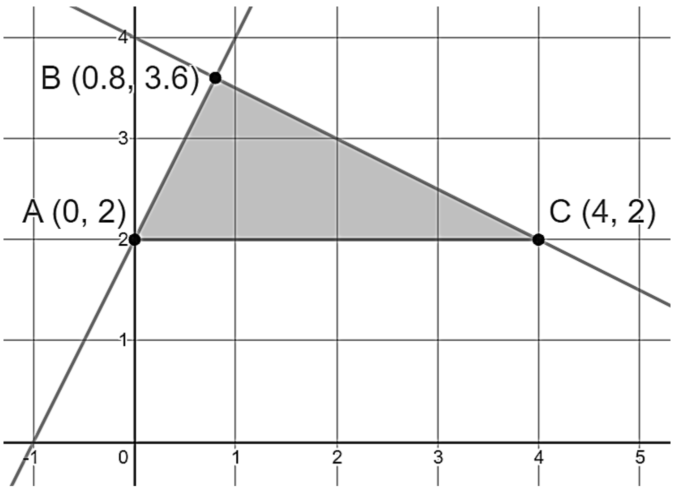


Practice Questions

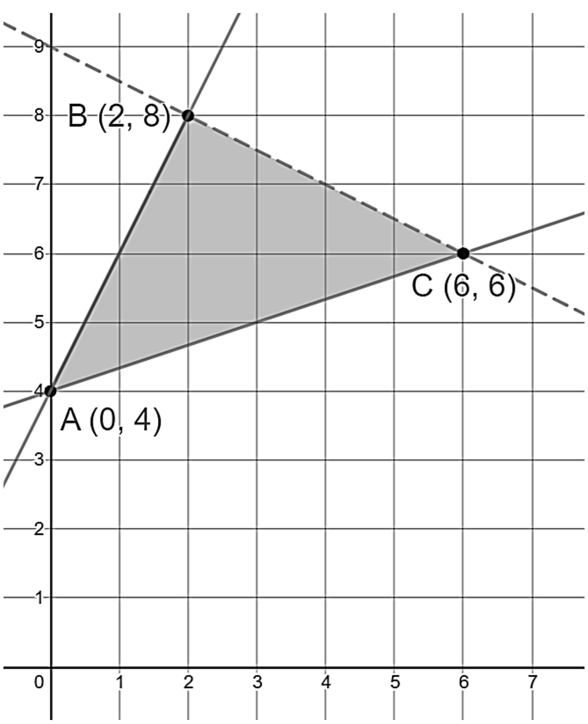
3) The polygon of constraints of an optimization question is represented on the graph below. The optimizing function is: $Profit = 3x + 6y$. How many points maximize this situation? What are they?



4) The polygon of constraints of an optimization question is represented on the graph below. The optimizing function is: $Cost = \$4x + \$2y$. What is the minimum cost in this scenario?

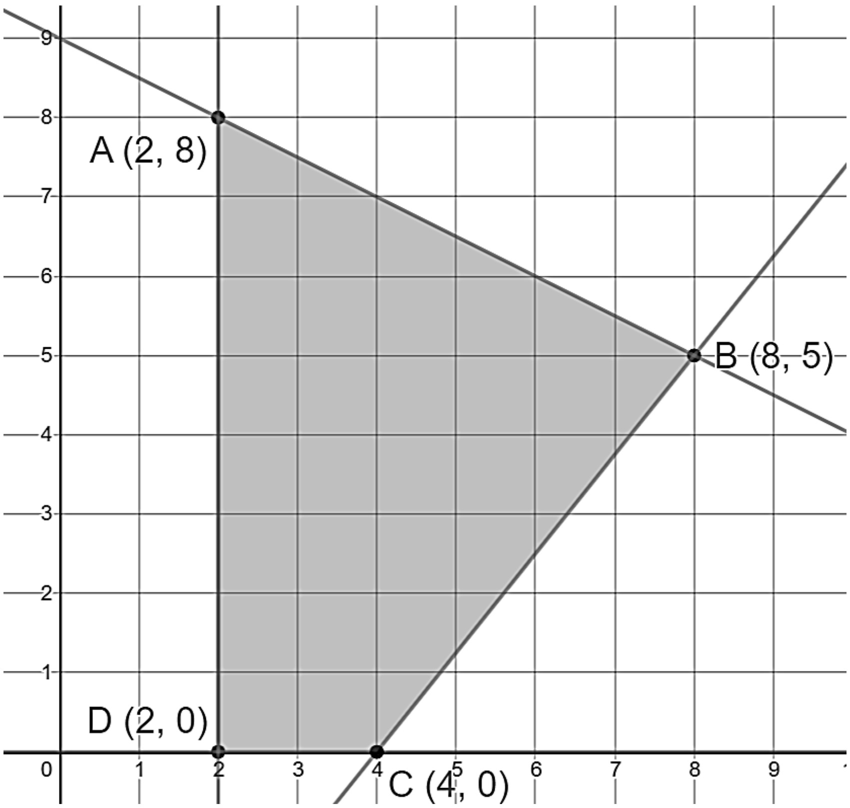


5) The polygon of constraints of an optimization question is represented on the graph below. The optimizing function is $Value = \$2x + \$10y$. What is the maximum value in this scenario?



6) The polygon of constraints of an optimization question is represented on the graph below. The optimizing function is $Value = 4x + 8y$.

What is the maximum value and how many points maximize the scenario?



1.10 OPTIMIZATION EXAM STYLE QUESTIONS

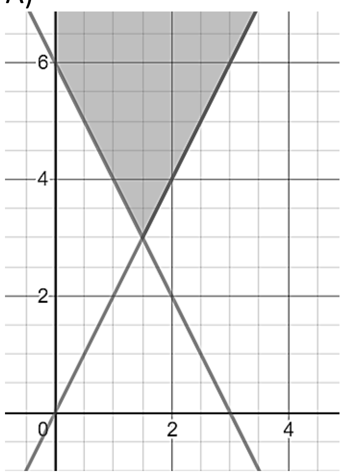
Multiple Choice

1) The system of inequalities below represents the constraints associated with an optimization situation.

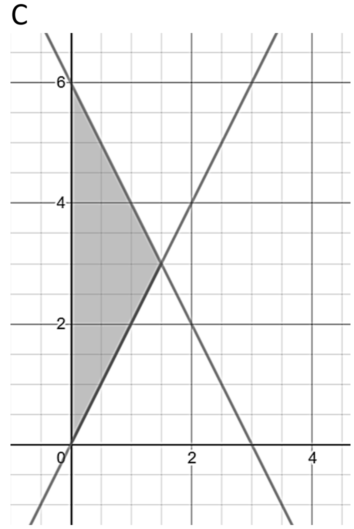
$$y \geq 0$$
$$x \geq 0$$
$$y \geq 2x$$
$$4x + 2y \leq 12$$

Which of the following represents the solutions for this system of inequalities?

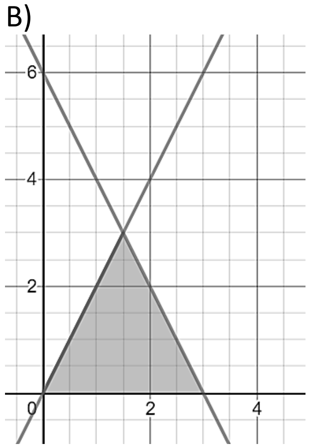
A)



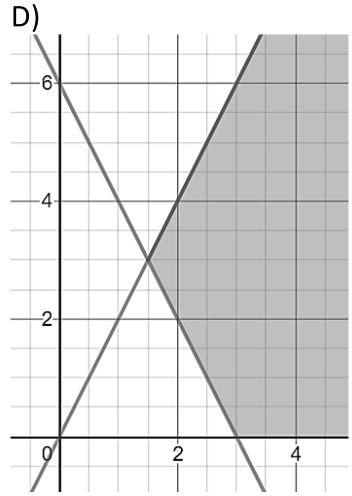
C



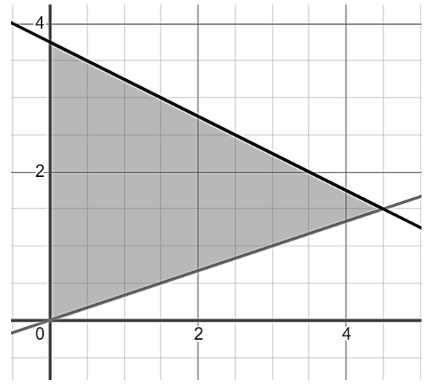
B)



D)



2) The graph below represents the polygon of constraints associated with an optimization situation.



Which of the following system of inequalities corresponds with this optimization system?

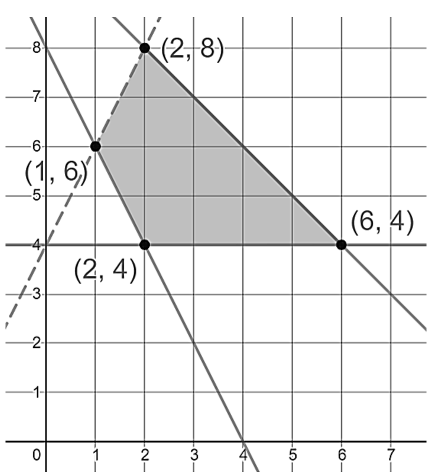
A)
$y \geq 0$
$x \geq 0$
$x \geq 3y$
$2x + 4y \leq 15$

C)
$y \geq 0$
$x \geq 0$
$x \leq 3y$
$2x + 4y \geq 15$

B)
$y \geq 0$
$x \geq 0$
$x \leq 3y$
$2x + 4y \leq 15$

D)
$y \geq 0$
$x \geq 0$
$x \geq 3y$
$2x + 4y \geq 15$

3) The polygon of constraints below is associated with an optimization situation.
The optimizing function is $Z = 100x + 100y$



How many points on the graph maximize the situation?

A) 1

C) 4

B) 2

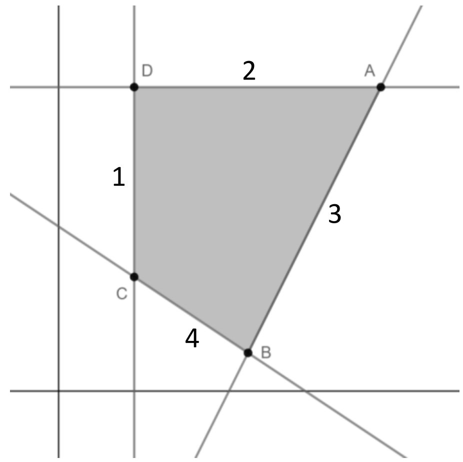
D) 5

Optimization Unit – Exam Style Questions

Short Answer

4) The constraints associated with an optimization situation are represented by the systems of inequalities and the polygon of constraints shown below. Each side of the polygon and its corresponding inequality are identified by the same number.

1) $x \geq 2$
2) $y \leq 8$
3) $y \geq 2x - 9$
4) $2x + 3y \geq 13$



What are the coordinates of vertex B of this polygon of constraints?

5) A high school is selling t-shirts and hoodies as a fundraiser for the class trip.
x: the number of t-shirts sold
y: the number of hoodies sold

Translate the following statements into inequalities.

- **The school will sell a maximum of 200 items.**
- **There will be at least twice as many t-shirts as hoodies sold.**

6) Each year Grade 11 students hold a car wash as a fundraiser for Prom. The revenue raised from the car wash is represented by the optimizing function $Z = 8x + 10y$

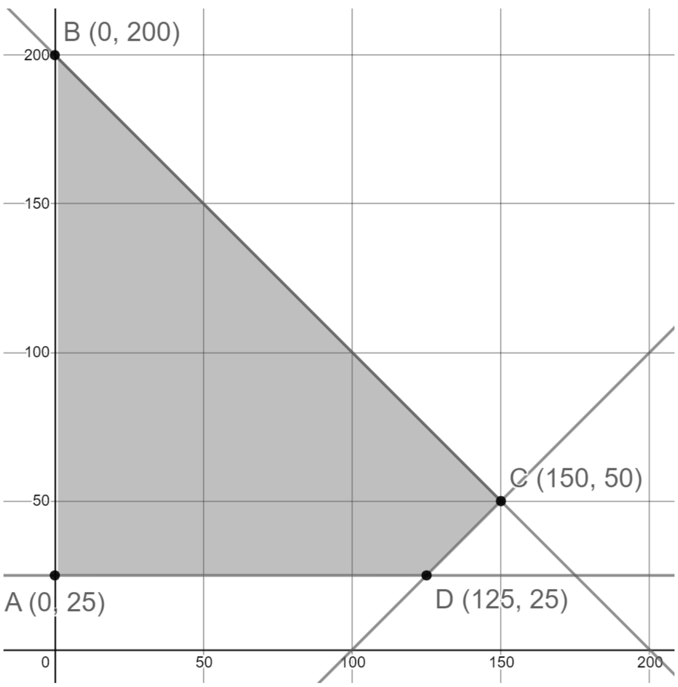
where: x is the number of cars washed

y is the number of trucks washed

The polygon of constraints below represents the combination of cars and trucks washed in a typical year.

This year, there is heavy construction near the car wash site. The organizers expected there will be fewer cars and trucks to wash as a result. The decrease in vehicles is represented by the inequality below.

$$x + y \leq 150$$



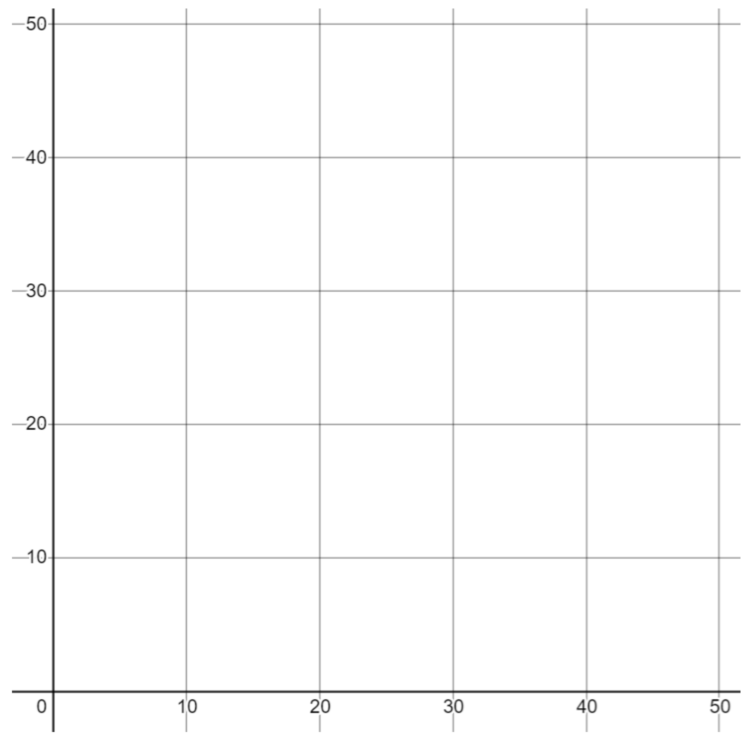
Vertex	Revenue $Z = 8x + 10y$
A (0, 25)	$250
B (0, 200)	$2000
C (150, 50)	$1700
D (125, 25)	$1250

By how much will the students' expected maximum revenue at the car wash decrease as a result of the construction?

Long Answer

7) Mackenzie is selling tulip bulbs as part of the school band fundraiser. They will offer a choice between standard red tulip bulbs and special orange tulip bulbs. Mackenzie's sales are limited by the following constraints:
   - Mackenzie can sell a maximum of 40 bulbs per day.
   - Mackenzie must sell a maximum of 10 red bulbs per day.
   - Mackenzie sells at least twice as many orange bulbs as red bulbs.

   Given that Mackenzie earns $1 for every red bulb sold and $2 for every orange bulb sold, what is the maximum profit Mackenzie can earn each day?



8) A company sells two different products. The first item is soap. The second item is lotion. Information about the sales of both items is below.

Soap

The company sells soap in two varieties: bars of soap and bottles of liquid soap. The soap sales must fit within the constraints given below:

- The company sells a maximum of 40 soaps per day.
- At least 5 bars of soap are sold each day.
- No fewer than 5 bottles of liquid soap are sold each day.
- The company sells no more than 3 times as many bars of soap as bottles of liquid soap

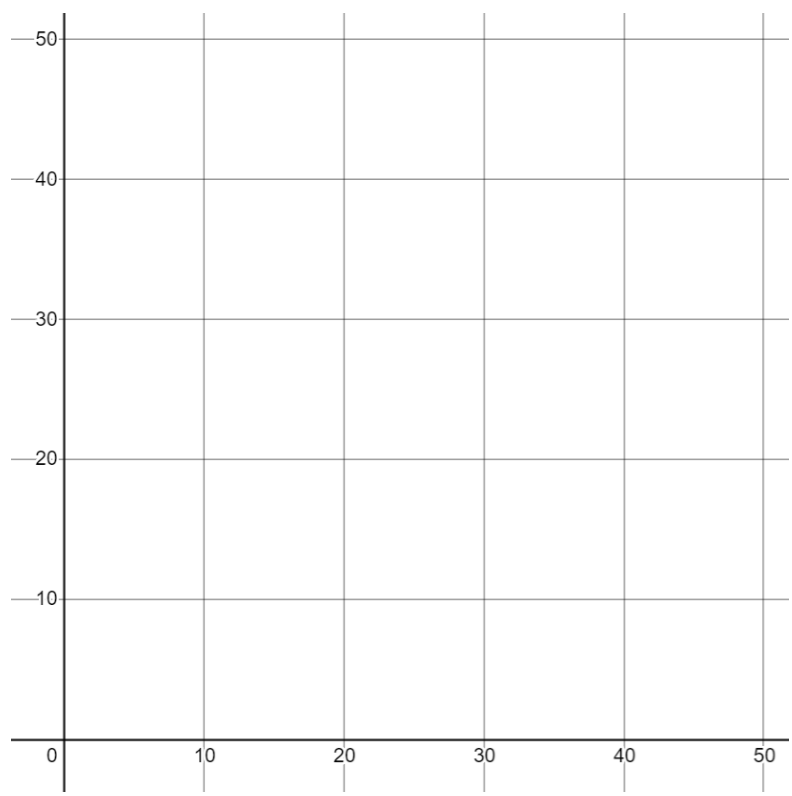
The company earns a profit of $2 per bar of soap and $3 per bottle of liquid soap.

Lotion

The company also sells two types of lotion: hand lotion and body lotion. Every day they sell 20 bottles of hand lotion and 25 bottles of body lotion.

The company earns the same profit on the total sale of lotion as the maximum profit from the sale of soap.

Given that they earn a profit of $3 on each bottle of hand lotion, what is the profit earned on each bottle of body lotion?



Optimization Unit – Exam Style Questions

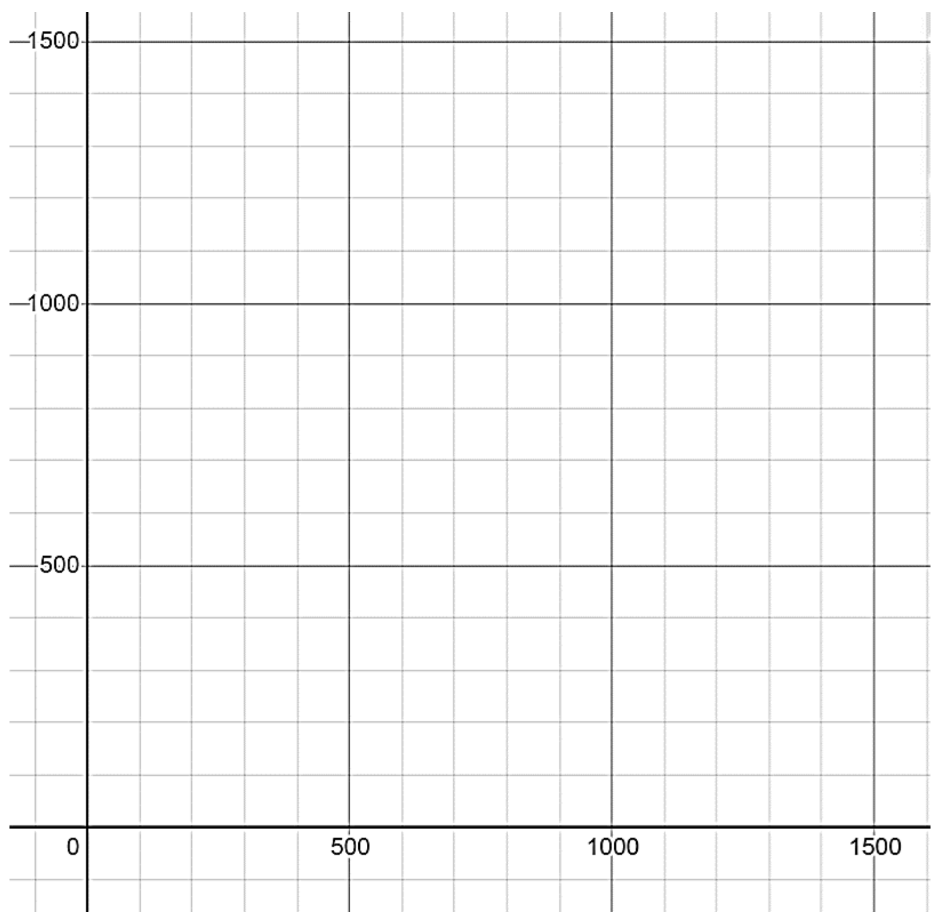
9) A company owns a vehicle manufacturing plant which has two assembly lines. One of the lines makes cars and one of the lines makes trucks. On a typical day, the plant must adhere to the following constraints in manufacturing cars and trucks.

- The plant can make no more than 1200 vehicles per day.
- The assembly line making cars can produce a minimum of 200 cars per day.
- The assembly line making trucks can produce no more than 800 cars per day.
- The manufacturing plant makes no more than twice times as many cars as trucks.

The company makes a profit of $2000 per car and $2500 per truck.

Today the manufacturing plant experienced a breakdown in the assembly lines and as a result the production capacity has been reduced. Today, the plant can make no more than 900 vehicles.

By how much did the breakdown reduce the maximum profit when compared to a typical day?



Optimization Unit – Exam Style Questions

Graph Theory Unit – Definitions

2.1 GRAPH THEORY DEFINITIONS

In graph theory, a graph is a collection of dots and lines. The lines show relationships that exist between the elements of a set (the dots).

Ex: An airline has flights to 5 different airports (Ottawa, Halifax, Toronto, Calgary, and Edmonton). The following is information about flights offered:

- From Ottawa, there are flights to Halifax and Toronto
- From Halifax, there are flights to Ottawa and Edmonton
- From Toronto, there are flights to Ottawa, Calgary, and Edmonton
- From Calgary, there are flights to Toronto and Edmonton
- From Edmonton, there are flights to Toronto, Halifax and Calgary

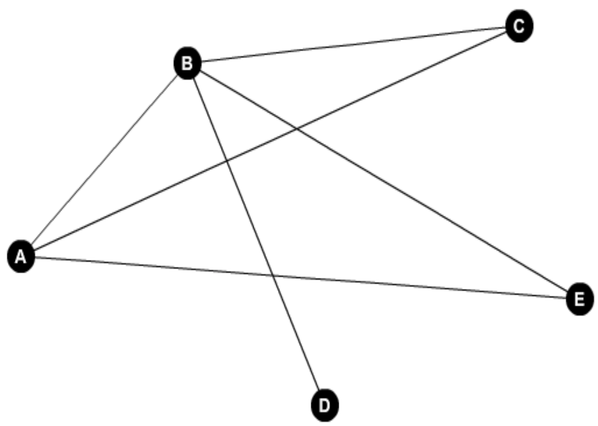
Draw a graph to represent this scenario.

Practice Question

1) A company is installing a new computer network and some computers have to be directly connected to others.

- There are 6 computers (A, B, C, D, E, F)
- A must be directly connected to B, C, E, F
- B must be directly connected to A, C and D
- C must be directly connected to A, B
- D must be directly connected to B and E
- E must be directly connected to A and D
- F must be directly connected to A

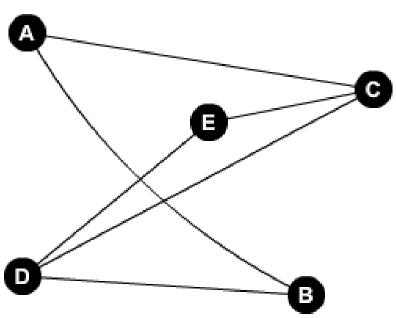
Draw a graph to represent this scenario.



- Vertex – a dot on the graph. The vertices are the elements of the set (people, places, etc.). They are typically labelled with letters.
- Edge – a line on the graph. The edges represent relationships between the vertices. Edges are labelled using the letters of the two vertices they connect. It does not matter which vertex is listed first.
- Order – the number of vertices in a graph.
- Degree – the number of edges that touch the vertex.

Practice Question

2) In the graph below, identify all the vertices and determine the degree of each vertex, identify all edges, and determine the order of the graph.

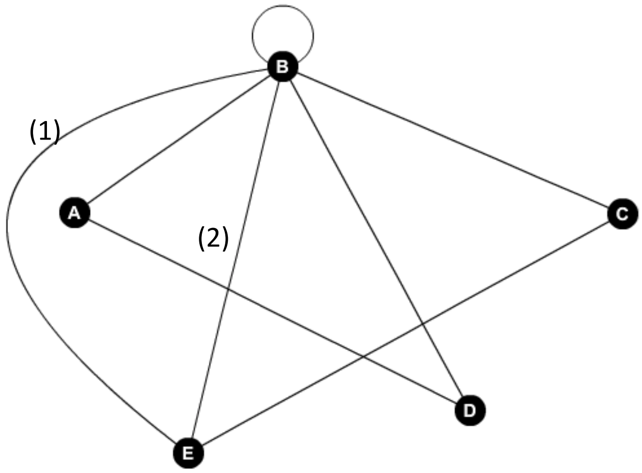


Vertices (and degree of each):

Edges:

Order:

Graph Theory Unit – Definitions

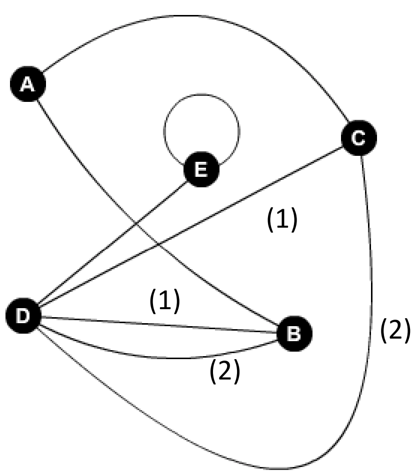


- Loop – an edge that starts and ends at the same vertex.

- Parallel edges – when two or more edges connect the same two vertices.

Practice Question

3) In the graph below, identify any loops and parallel edges

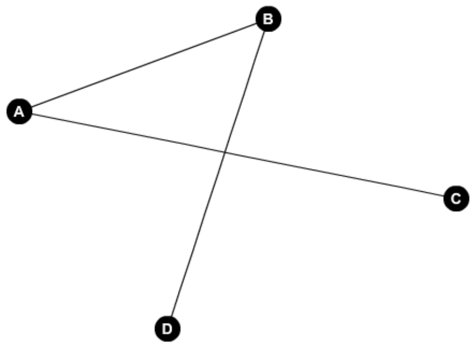


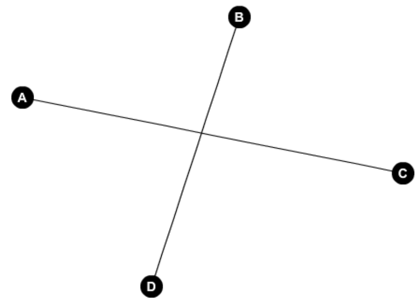
Loop(s):

Parallel Edge(s):

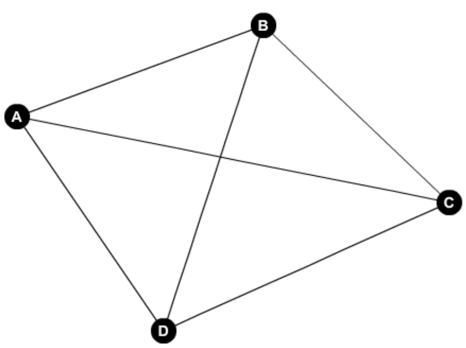
Graph Theory Unit – Definitions

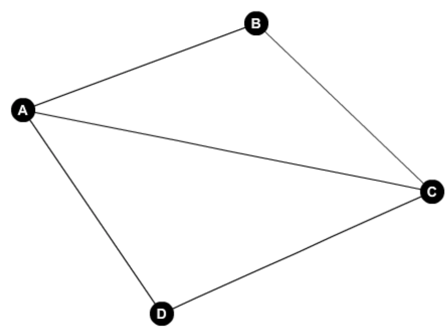
- A graph is **connected** when each vertex is connected to every other vertex by an edge *or by a series of edges*. This means that you can start at one vertex and get to every other vertex.





- A graph is **complete** when an edge connects every pair of vertices. This means there is only one edge separating every pair of vertices.

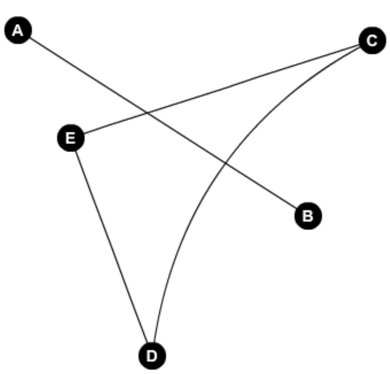




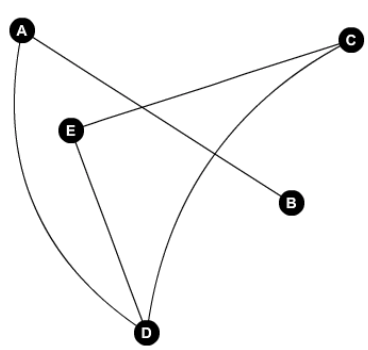
Practice Question

4) Determine if the graphs below are connected, complete, or neither.

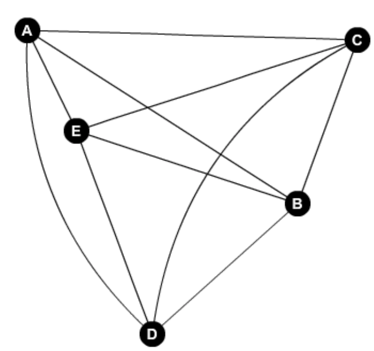
a)



b)

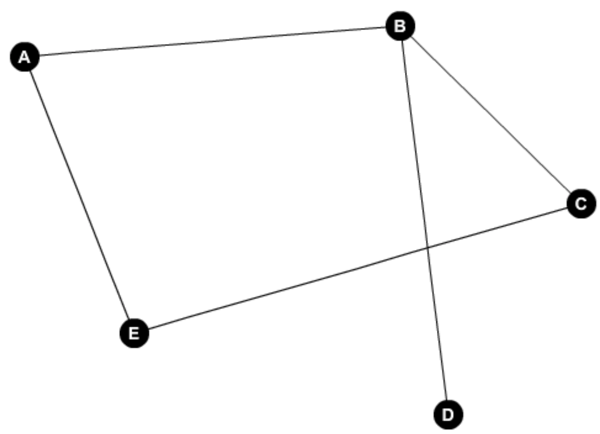


c)



Graph Theory Unit – Definitions

- The **complement** of a graph is a graph that contains the edges necessary to complete the initial graph. You can think about this as the “missing edges”.

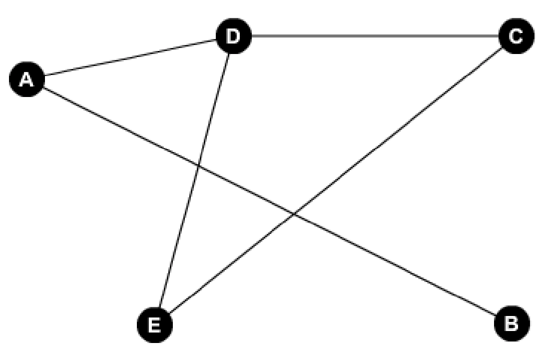


A graph

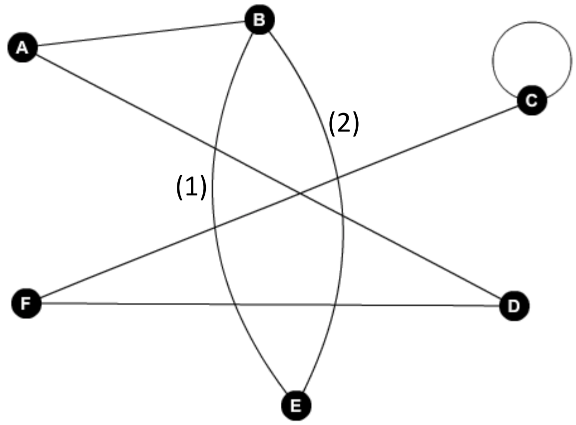
Draw the complement

Practice Questions

5) Create the complement of the graph given below.



6)



a) List all vertices and the degree of each

b) List all edges

c) Determine the order of the graph

d) Identify any loops

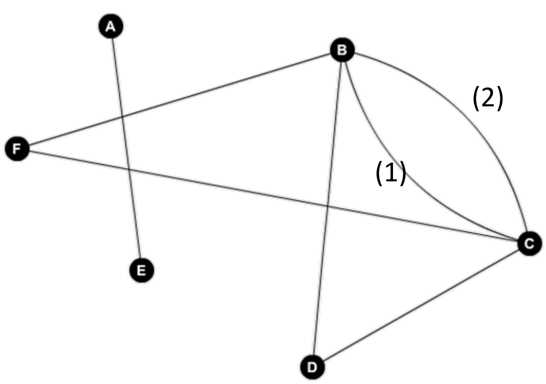
e) Identify any parallel edges

f) Determine whether the graph is connected

g) Determine whether the graph is complete

h) Draw the complement of the graph

7)



a) List all vertices and the degree of each

b) List all edges

c) Determine the order of the graph

d) Identify any loops

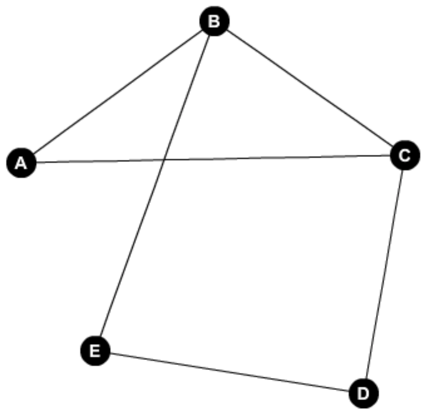
e) Identify any parallel edges

f) Determine whether the graph is connected

g) Determine whether the graph is complete

h) Draw the complement of the graph

2.2 GRAPH THEORY PATHS AND CIRCUITS



- A **path** on a graph is how we can get from one vertex to another. Vertices and edges can be repeated.
  On the graph above, an example of a path is

- A **simple path** is a path where no edges are repeated (vertices can be repeated).
  On the graph above, an example of a simple path is

- The **length** of a path is the number of edges it contains.
  On the graph above, the length of path ABCDEBAC is:

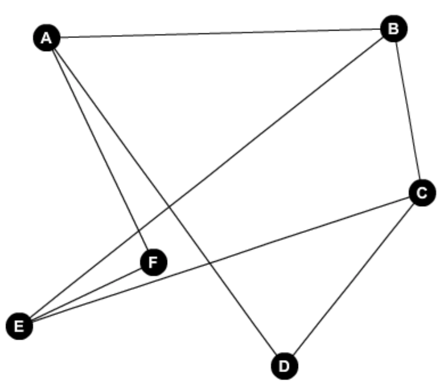
  On the graph above, the length of path CBEDCA is:

- The **distance** between two vertices is the length of the shortest path joining the vertices.
  On the graph above, $d(A, E)$ is:

  On the graph above, d(A,C) is:

Practice Question

1) On the graph below:



a) Name a path

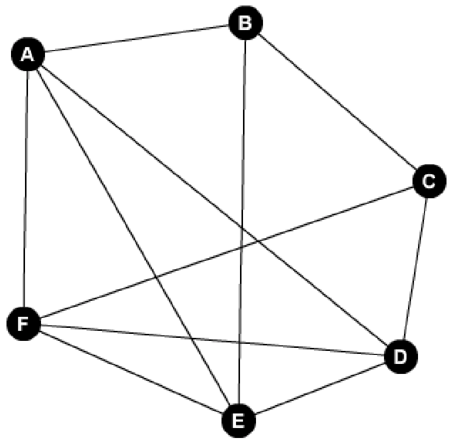
b) Name a simple path

c) What is the length of path DABEFAB?

d) Name a simple path with length 4

e) What is the distance between C and A?

f) What is the distance between A and D?



- A **circuit** is a path that starts and ends at the same vertex.

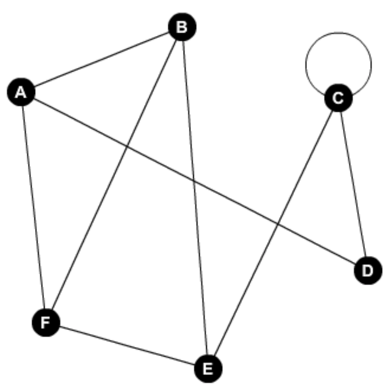
On the graph above, an example of a circuit is:

- A **simple circuit** is a circuit that has no repeated edges (but it does not need to use every edge)

On the graph above, an example of a simple circuit is:

Practice Questions

2) On the graph below:



a) find a path with length 4

b) find a simple path with length 6

c) find a circuit with length 5

d) find a simple circuit with length 3

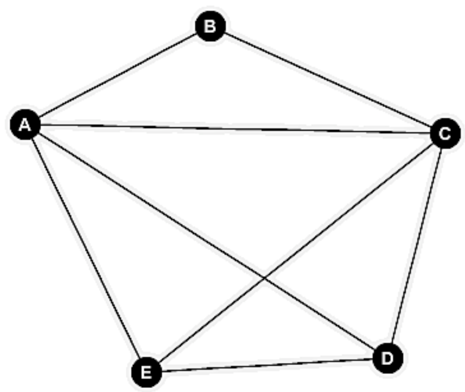
e) find the distance between vertices A and C

Graph Theory Unit – Euler Path and Circuit

2.3 EULER PATH AND CIRCUIT

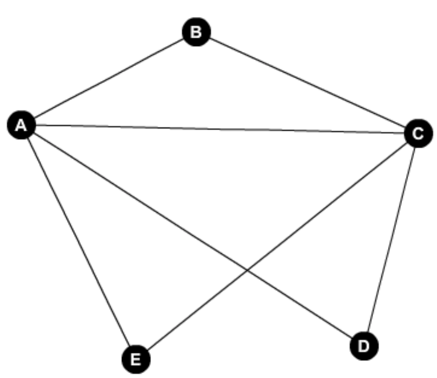
- An **Euler path** (pronounced "oiler") is a path that travels over every edge once and only once.
  - An Euler path exists if exactly 2 vertices have a degree that is an odd number (and the rest have a degree that is an even number).
    - The Euler path will start at one of the vertices with an odd degree and end at the other vertex with an odd degree.
  - An Euler path also exists if all vertices have a degree that is an even number (but this is a special case, called an Euler circuit – see below)

Ex: Find an Euler path in the graph below



- An **Euler circuit** is a circuit that travels over every edge once and only once (it is an Euler path that starts and ends at the same vertex).
  - An Euler circuit is a special case of an Euler path. It is an Euler path that begins and ends at the same vertex.
  - An Euler circuit exists if all vertices have degrees that are even numbers.

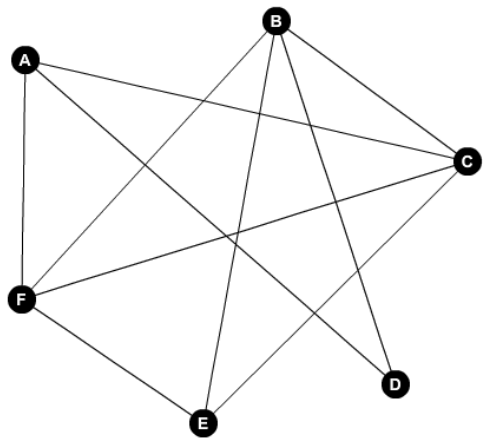
Ex: Find an Euler circuit in the example below



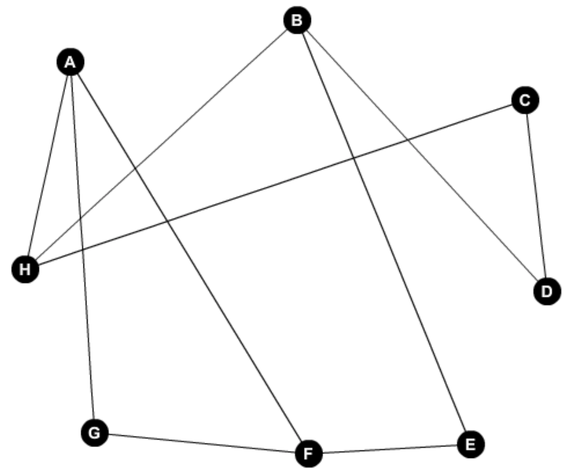
Practice Questions

1) Does an Euler path exist in the graphs below? If yes, identify one.

a)

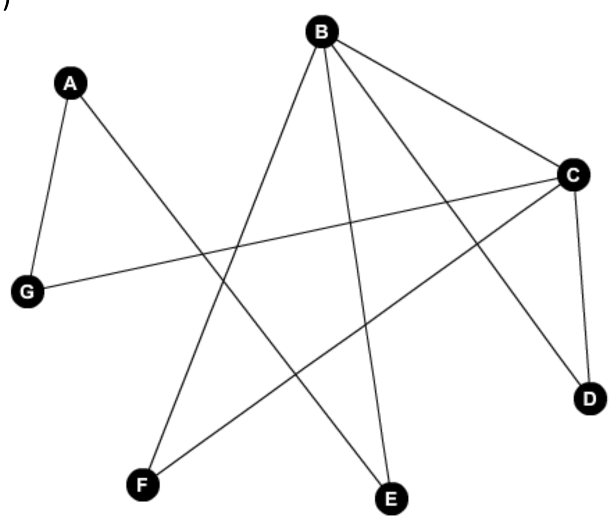


b)

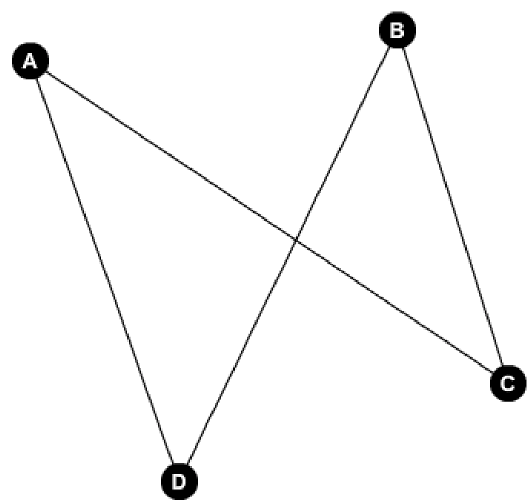


2) Does an Euler circuit exist in the graphs below? If yes, identify one.

a)



b)

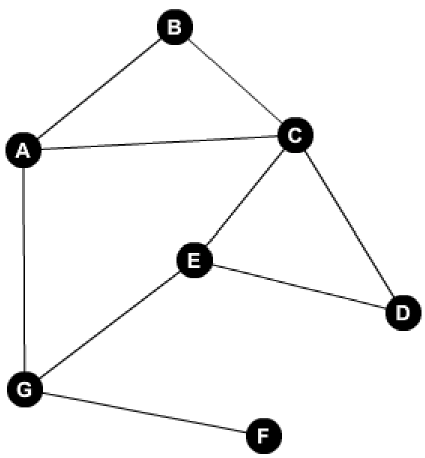


Graph Theory Unit – Hamiltonian Path and Circuit

2.4 HAMILTONIAN PATH AND CIRCUIT

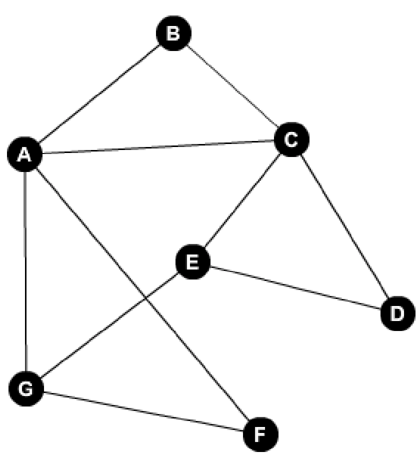
- A **Hamiltonian path** is a path that passes through every vertex once and only once (it does not need to include every edge).
    - There is no good way to determine whether a Hamiltonian path exists other than trying to find one.

Ex: Find a Hamiltonian path in the graph below



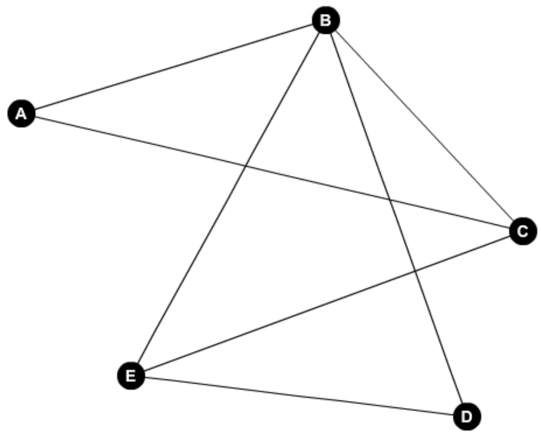
- A **Hamiltonian circuit** is a Hamiltonian path that begins and ends at the same vertex (so the first vertex is repeated as the last vertex, but no other vertex is repeated and every other vertex is included).

Ex: Find a Hamiltonian circuit in the graph below

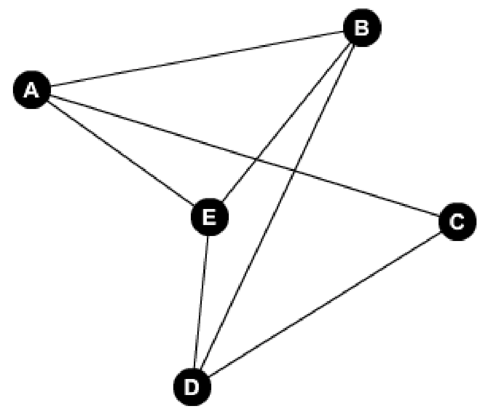


Practice Questions

1) Identify a Hamiltonian path in the graph below.

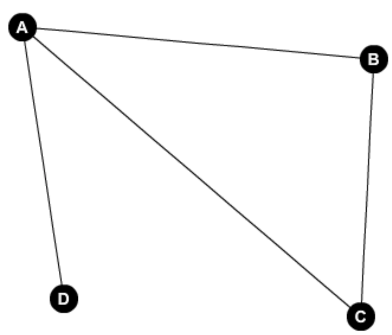


2) Identify a Hamiltonian circuit in the graph below.

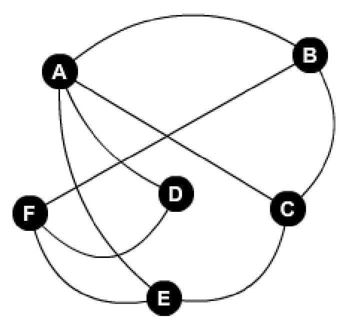


3) In each of the following graphs, indicate whether the following exist:
i) Euler path; ii) Euler circuit; iii) Hamiltonian path; iv) Hamiltonian circuit

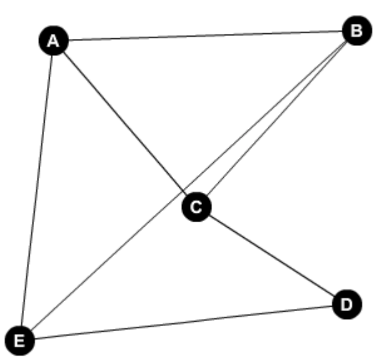
a)



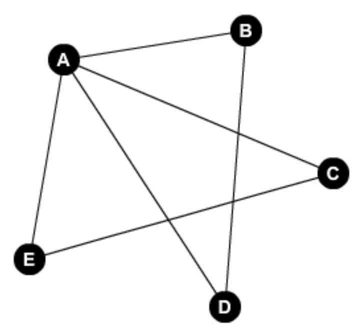
c)



b)



d)

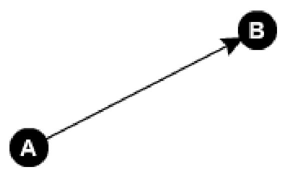


Graph Theory Unit – Directed Graphs

2.5 DIRECTED GRAPH

- A **directed edge** is like a one-way street – we can only move along that edge in one direction. It is represented with an arrow on the edge.
- A **directed graph** has one or more directed edges.

Ex:

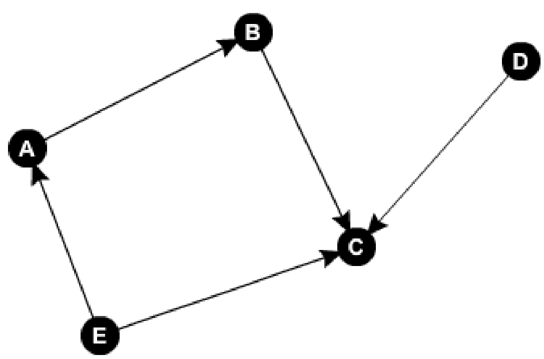


We can go from vertex A to vertex B, but not from vertex B to vertex A.

When we name directed edges, order matters.

This edge is AB (and not BA)

Ex:

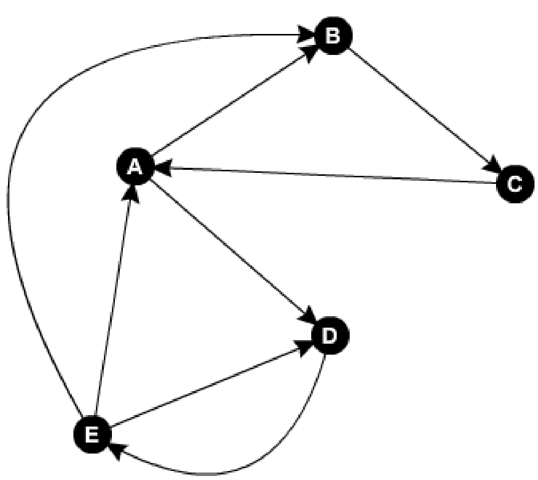


In this example, a path is ABC, but not ABCD.

This example does not have any circuits.

Practice Questions

1) In the graph below:



a) Identify a path of length 4.

b) Identify a simple circuit of length 3.

c) Does path A-E-D exist? Explain.

d) What is the length of path A-B-C-A-D-E?

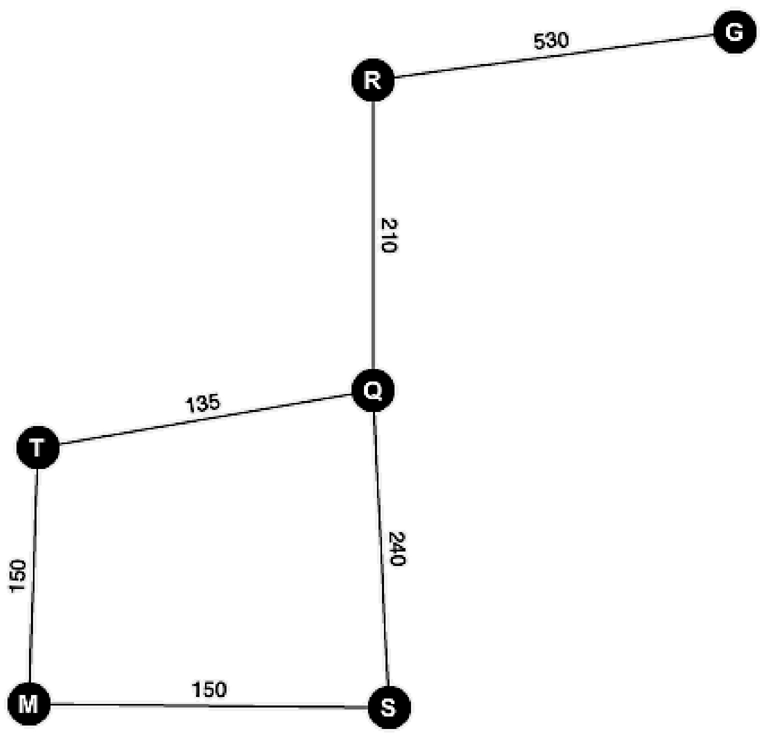
e) Determine $d(A, C)$.

Graph Theory Unit – Weighted Graphs

2.6 WEIGHTED GRAPHS (NETWORKS)

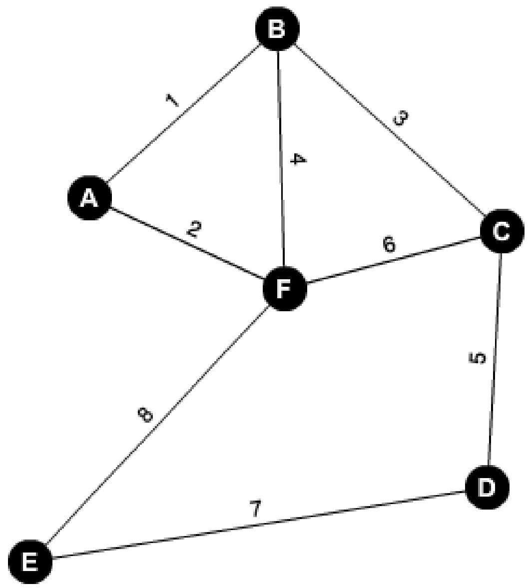
- A **weight** is a number, which could represent distance, time, etc.
- A **weighted graph** is a graph in which every edge is assigned a weight.
- The **weight of a path** is found by adding the weights of all the edges included in the path.
- A weighted graph can be directed or not.

Ex: The network below shows the distance (in km) between towns linked by a rail network.
Transporting heavy equipment by train costs approximately $100 per km.
Give the approximate cost of transporting heavy equipment from G to M if the train takes the shortest route.



Practice Question

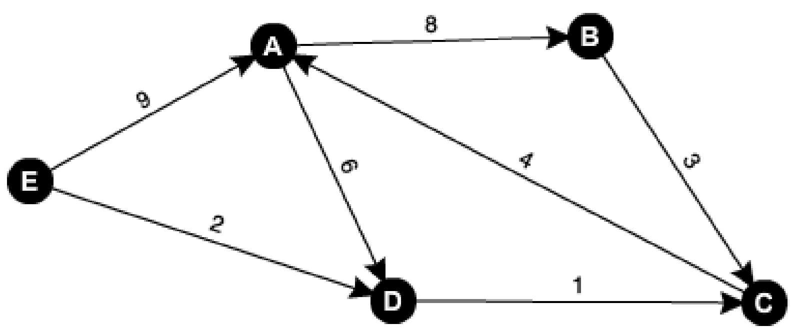
1) Based on the graph below, determine:



a) The weight of path A-B-C-F

b) The weight of circuit C-D-E-F-C

2) The graph below is weighted and directed.



a) Determine the weight of path E-D-C-A-D

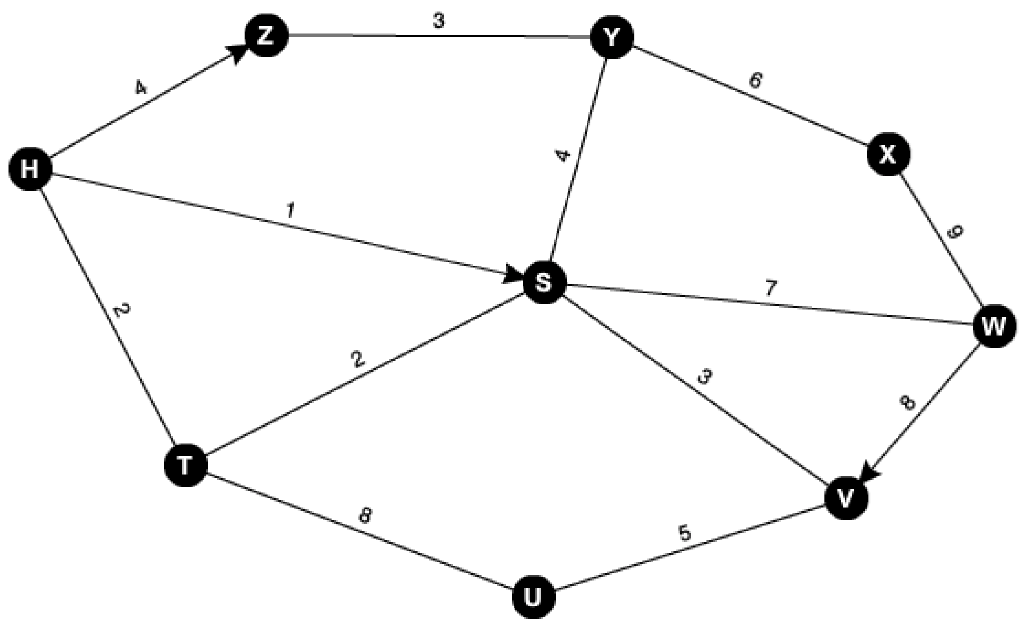
b) Determine the weight of path B-C-A-D

c) Determine the weight of path E-A-B-C

d) Identify all the simple paths going from vertex E to vertex B

e) Of the paths identified in part d, which one has the smallest weight?

3) The following graph presents the length (in km) of various routes Louis could take as he trains for a marathon. He begins and ends each training session at his home, H.



a) For easy runs, Louis wants to run less than 10km without taking the same road twice. What route can he take and how long is his run?

b) How long is Louis's run if he takes the following route: H-Z-Y-S-V-U-T-H?

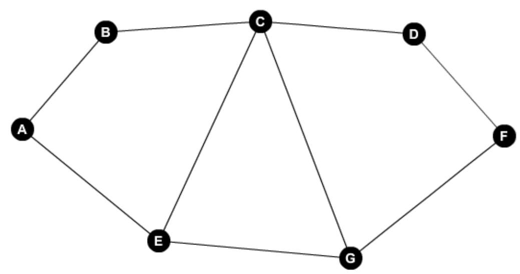
c) Louis would like to run a circuit that covers between 30 and 35 km without taking the same road twice. Is this possible (and if yes, what route could he take)?

2.7 PATH OF OPTIMAL VALUE

- When we want to find the **path of optimal value**, we are looking for the longest or shortest path (the maximum or minimum, depending on the question) from one vertex to another.
  - On a graph without weighted edges, the **path of minimum value** between two vertices corresponds to the path using the fewest edges between those vertices.
  - On a graph without weighted edges, the **path of maximum value** between two vertices corresponds to the simple path (no repeated edges) using the most edges between those vertices.
  - On a weighted graph, the **path of minimum value** between two vertices corresponds to the path that has the smallest weight between those vertices.
  - On a weighted graph, the **path of maximum value** between two vertices corresponds to the simple path (no repeated edges) with the largest weight between those vertices.

In the graphs below, determine the path of minimum value and the path of maximum value with an initial vertex A and a final vertex F.

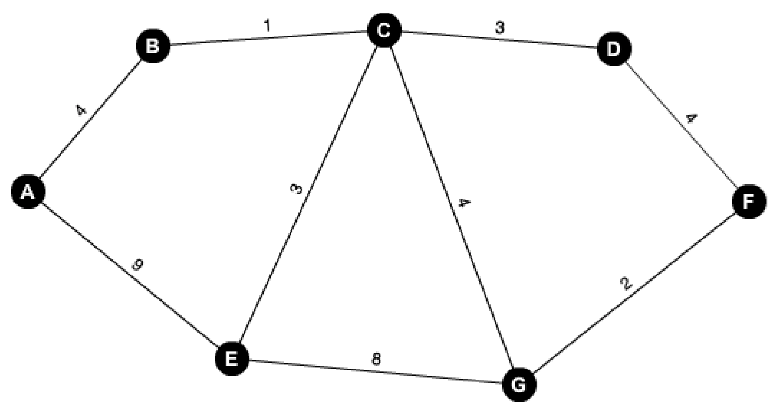
a)



Path of minimum value:

Path of maximum value:

b)

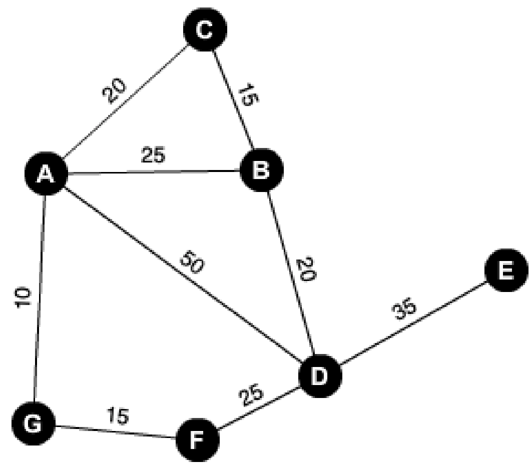


Path of minimum value:

Path of maximum value:

Practice Question

1) In the graph below, the values represent the amount (in thousands of dollars) that a construction company must pay to transport its equipment from one city to another.



a) Determine the minimum transport cost from City A to City E

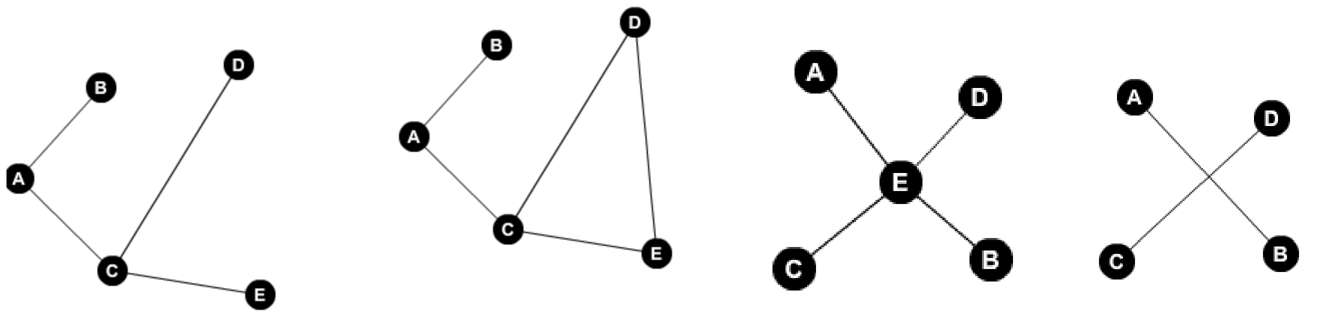
b) Determine the minimum transport cost from City G to City B

Graph Theory Unit – Tree of Optimal Value

2.8 TREE OF OPTIMAL VALUE

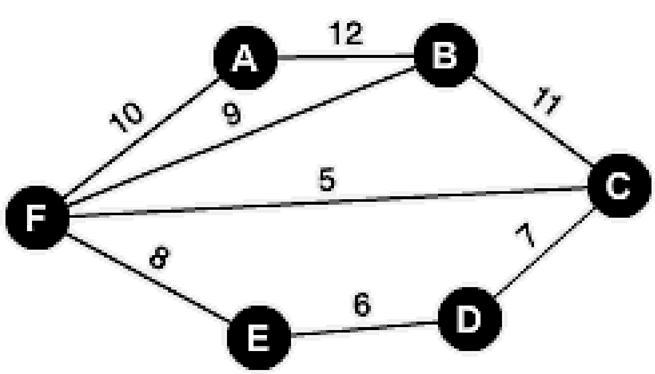
- A **tree** is a connected graph with no simple circuits

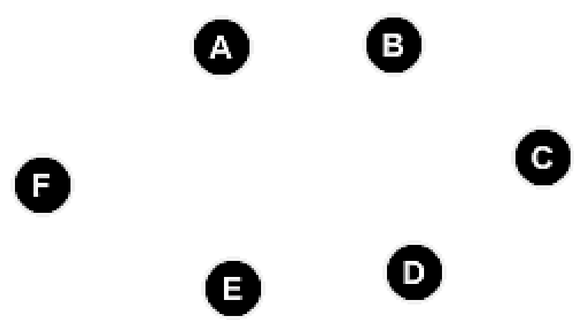
Examples:



- A **tree of optimal value** selects only some edges in a graph such that all vertices are connected, there are no simple circuits, and we use either the largest or smallest weighted edges.
- There are several steps to creating a tree of optimal value. If we are looking for a tree of minimum value:
    - Copy the vertices of the graph
    - Select the edge with the lowest value and draw it.
    - Of the remaining edges, select the one with the lowest value and draw it.
    - Keep selecting the smallest edge and drawing it, unless an edge will create a simple circuit. Then skip that edge and move to the next one. Once you have created a tree, stop drawing edges.
- To create a tree of maximum value, follow the steps above, but use edges with the largest value.

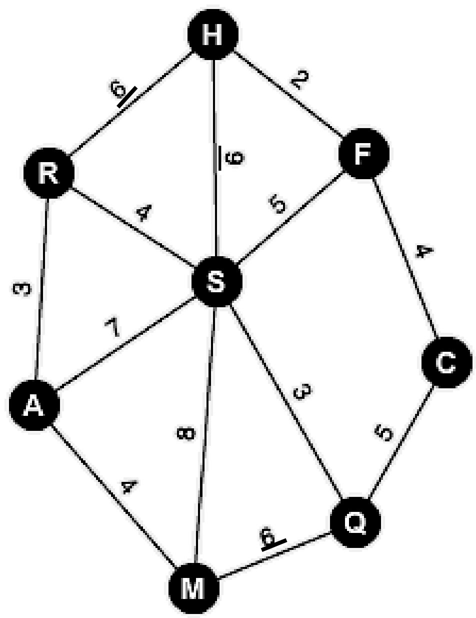
Ex: Create a tree of minimum value given the graph below





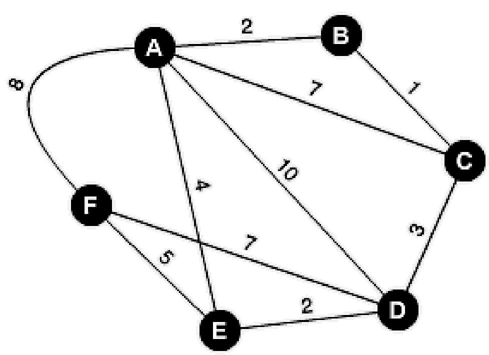
Practice Questions

1) A hydroelectric dam generates electricity which is transmitted using high-tension lines to Quebec's main urban centers. These transmission lines lose some of their power as the electricity is being transmitted. The graph below shows some of these lines and the power loss in tens of megawatts. Vertex H is the hydroelectric dam.



The facility's engineering crew wants to minimize the power loss while connecting all the urban centers.
Which high-tension lines should be used and what is the total power loss?

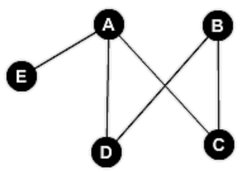
2) A city park has developed a series of trails between various parking areas. In the winter, the city wants to maintain the fewest number of trails such that all parking areas are still connected, either directly or indirectly, but the total trail distance is maximized. The graph below represents the parking areas (vertices) and trails (edges) where the weight of each edge represents the length (in km) of that trail.



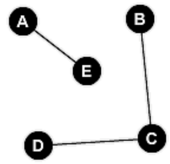
Which paths should the city maintain in the winter and what is the total distance of trails available?

3) In the following graphs, what edge(s) would need to be added or removed in order for the graph to be a tree?

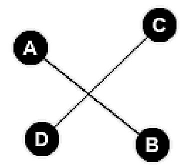
a)



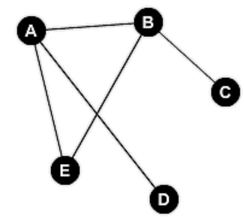
b)



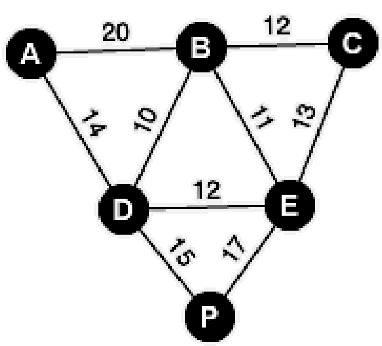
c)



d)

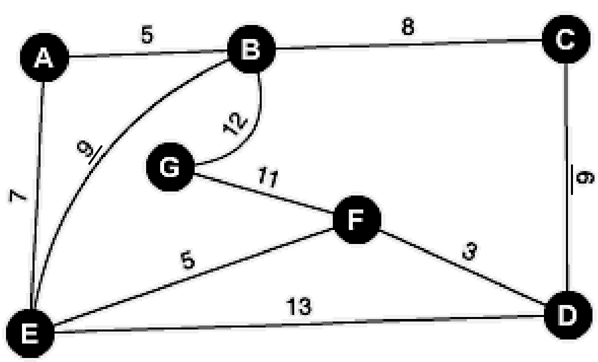


4) The sewage lines in a city need an upgrade and every neighborhood must be connected (either directly or indirectly) by sewage line to the purification plant, P. The graph below represents the neighborhoods in a city and the cost of upgrading the sewage lines (in thousands of $).



What is the minimum cost for the city to upgrade the sewage lines?

5) The cost of hiring a taxi is based on the amount of time it takes to get from your starting place to your destination. The graph below represents a taxi ride's travel time (in min) between intersections.



If a taxi costs $1.50 per minute, calculate:

a) The minimum cost to get from Point F to Point B

b) The minimum cost to get from Point A to Point D

Graph Theory Unit – Critical Path

2.9 CRITICAL PATH

If we have many tasks that are necessary to complete a project, and some of the tasks can be carried out simultaneously, we can use graph theory to determine the minimum amount of time necessary to complete the project.

- A **critical path** (the minimum amount of time to complete all steps) is the simple path of ***maximum value***.

There are 5 keys to determining a critical path

1. Each vertex is a step in the project.
2. Each edge is weighted with the time it takes to complete the step (using the vertex at the start of the line) and each edge is directed.
3. Use the steps and time to create a graph.
4. Find all possible simple paths from the start of the project to the end.
5. Choose the *longest* path.

Ex: There are several steps required to launch a business. The following table lists the necessary steps, the time it takes to complete each step, and provides information on whether there are any steps that need to be completed before a step can begin.

Step	Description	Execution Time (in days)	Prior Steps
A	Prepare a business plan	30	None
B	Conduct market research	10	A
C	Find partners	25	A
D	Find location	20	A
E	Analyze market research	5	B
F	Evaluate product-distribution system	15	C and D
G	Arrange financing	35	E and F
H	Launch company	None	G

Determine the minimum time necessary to complete all the steps.

Practice Questions

1) An accounting firm hires a company to review its computer system. The following is the company's proposal for the implementation of a new system.

Step	Description	Execution Time (days)	Prior Steps
A	Analysis of needs	10	None
B	Detailed analysis of project	8	A
C	Purchase of computers	21	B
D	Training of programming team	3	B
E	Designing the accounting system	6	C
F	Coding the accounting system	18	C
G	Updating the computers	5	E and F
H	Installation and delivery of the computers and accounting system	4	D and G
I	End of computer work	None	H

Determine the minimum amount of time necessary for the implementation of this new system.

2) Tyson is having friends over for dinner. He starts preparing the meal at 3:45 pm. The steps required for the dinner preparation are as follows:

Step	Description	Execution time (min.)	Prior Steps
A	Choose the menu	5	None
B	Peel the carrots	5	A
C	Prepare the meatballs	15	A
D	Prepare the sauce	5	B
E	Cook the meatballs in the sauce with the carrots	45	C and D
F	Prepare the salad	10	D
G	Prepare the appetizer	15	C
H	Cook the appetizer	15	G
I	Prepare the dessert	20	G
J	Set the table	10	E and F
K	Serve dinner	5	H, I, J
L	End of preparation	None	K

Tyson would like to complete all the steps by 5:00pm. Is this possible? Explain.

3) The process of writing, giving, and analysing tests teachers give to a class requires several steps, some of which can be carried out simultaneously. The steps required are as follows:

Step	Description	Time (days)	Prior Steps
A	Review the material	1	None
B	Decide on structure of exam	1	None
C	Write the questions	5	A and B
D	Review the questions	2	C
E	Write the exam	1	D
F	Check the exam	1	E
G	Give the exam	1	F
H	Correct the exam	2	G
I	Analyse the results	1	G
J	End of task	None	H and I

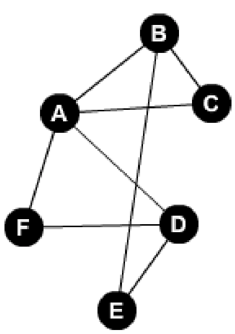
What is the minimum amount of time necessary to complete this project?

Graph Theory Unit – Chromatic Number

2.10 CHROMATIC NUMBER

- The **chromatic number** of a graph is the fewest number of colors required to color all the vertices of a graph while making sure no two adjacent vertices (vertices directly connected to one another) are the same color.
- Chromatic number can be used to:
  - Categorize information into as few groups as possible (ex: the number of colors required to color a map so that no countries sharing a border are the same color).
  - Categorize or group data based on incompatibilities (ex: making groups so that people who do not work well together are in different groups).
- There are 4 steps to determining the chromatic number of a graph
  1. Find the vertex with the highest degree and color it (if there is a tie, start with either)
  2. Find the vertices adjacent to the one you colored. Starting with the highest degree of those, color them a different color than the first vertex. Use the same color for all, unless they are adjacent. Use as few colors as possible.
  3. Of the remaining vertices, start with the highest degree and color is so that it is not the same color as any vertex it is adjacent to, using colors already used, if possible.
  4. The number of colors used is the chromatic number

Ex: Find the chromatic number of the graph below



Ex: You are the team leader and have a group of 5 people. You want to make small groups to complete a task. It does not matter if the groups are the same size, but you must make sure people who do not work well together are in different groups.

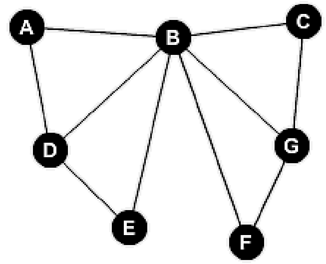
- Beth cannot work with Sue
- David and Jonas argue so will not get any work done
- Sue and Brian are unproductive together
- David and Sue are poorly matched
- Brian and David are not compatible

What is the fewest number of groups you can make?

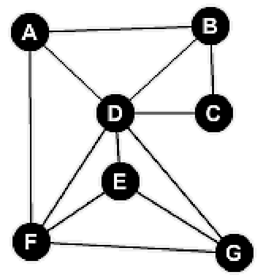
Practice Questions

1) Determine the chromatic number for the graphs below.

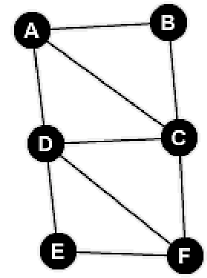
a)



b)



c)



2) Lydia has to color the map of South America for her geography class. The teacher has asked students not to use the same color for two countries that share a common border. Lydia only has 4 colors she can use and she wants to know if she can complete this task.

Using the map:

a) Construct a graph (vertices are the countries and edges represent the countries that share a common border).

b) Determine the chromatic number

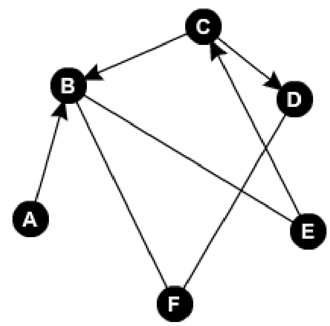
c) Can Lydia complete the task?

Graph Theory Unit – Exam Style Questions

2.11 EXAM STYLE QUESTIONS

Multiple Choice

1) Consider the graph below.



Which of the following does this graph contain?

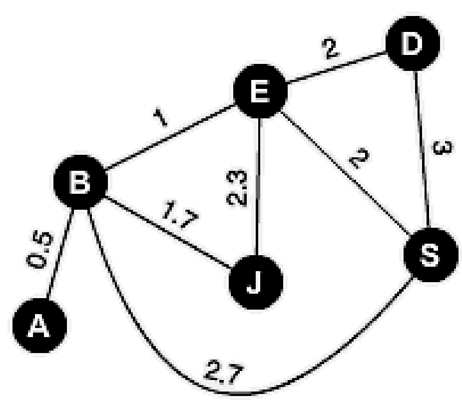
A) Euler Path C) Hamiltonian Path

B) Euler Circuit D) Hamiltonian Circuit

2) Emily (E), Jake (J), Sevanna (S), and Durham (D) are sharing a taxi to the airport (A). Vertex B is an intersection.

The taxi will start at Emily's house and then will pick up Jake and Sevanna in any order.
They will then drive to pick up Durham before heading to the airport.

The graph below represents a map of the possible routes the taxi can take with the weights representing the distance (in km). The taxi can use the same edge more than one.



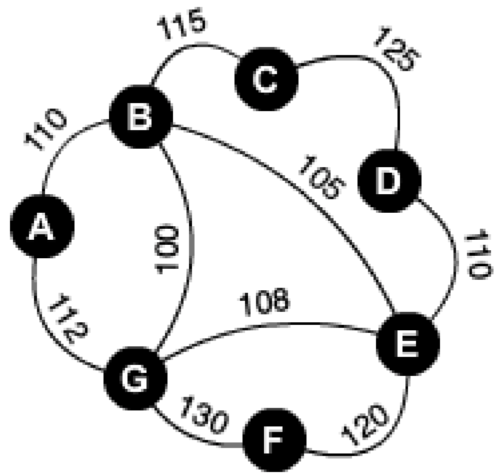
What is the length of the shortest route the taxi can take?

A) 10.5 km C) 13.1 km

B) 12.6 km D) 14.1 km

3) Roger wants to install a watering system in his garden. On the graph below:

- The sprinklers are represented by vertices A, B, C, D, E, F, and G.
- The pipes connecting the sprinklers are represented by the edges.
- The number associated with each edge represents the cost of installation, in dollars.



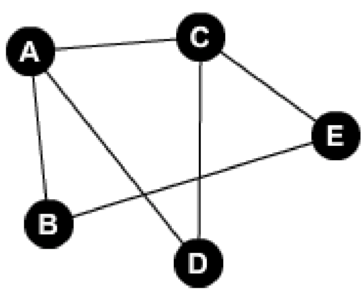
What is the lowest cost to connect all the sprinklers while minimizing the cost?

A) $313   C) $822

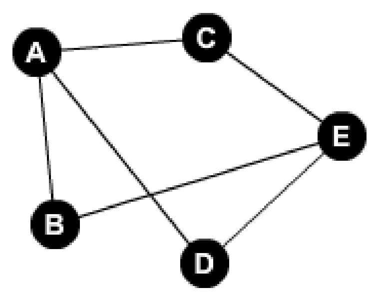
B) $660   D) $880

4) **Which of the following graphs contains and Euler circuit?**

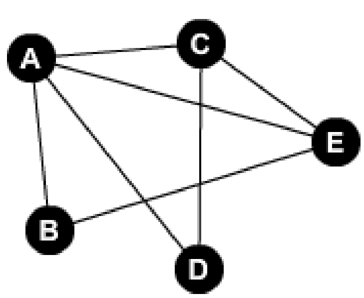
A)



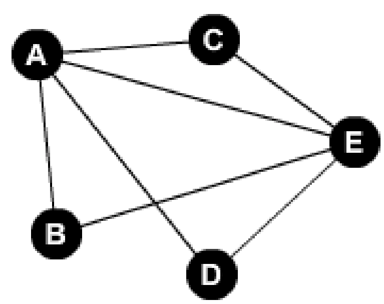
C)



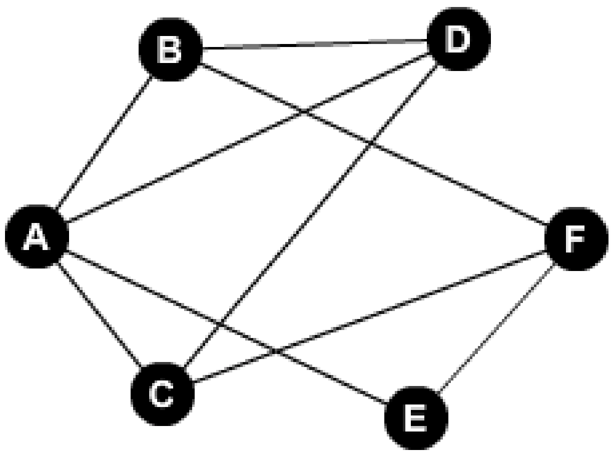
B)



D)



5) **What is the chromatic number for the graph below?**



A) 2

C) 4

B) 3

D) 5

Short Answer

6) Dr. James is dividing her class into groups and she wants to make sure all students in a group work well together. Dr. James wants to create the fewest number of groups and wants to make sure every student is in a group with at least one other student.

The table below contains information about students who do not work well together.

Student	Is incompatible with…
Amanda	Chris, David, Francis
Bob	David, Francis
Chris	Amanda, Eli, Francis
David	Amanda, Bob
Eli	Chris, Francis
Francis	Amanda, Bob, Chris, Eli

What is the fewest number of groups Dr. James can make and which students are grouped together?

Short Answer

7) Sierra is preparing a dinner for her friends. The following table shows the different steps involved in preparing and serving the dinner.

Tasks	Time (min.)	Prerequisite
A. Mix brownies	15 min.	none
B. Bake brownies	45 min.	A
C. Chop the vegetables	10 min.	none
D. Season the vegetables	5 min.	C
E. Prepare the chicken	15 min.	none
F. Cook the chicken	90 min.	E
G. Peel the potatoes	10 min.	none
H. Boil the potatoes	15 min.	G
I. Mash the potatoes	5 min.	H
J. Remove the chicken from oven to cool	10 min.	F
K. Roast the vegetables	15 min.	D, J
L. Serve the meal	None	B, I, K

What is the minimum amount of time Sierra needs to prepare and serve the dinner?

Graph Theory Unit – Exam Style Questions

Long Answer

8) A company has taken on a new project and the table below shows the steps necessary to complete this project.

Tasks	Time (days)	Prerequisite
A.	4	none
B.	5	A
C.	6	A
D.	4	B
E.	10	B
F.	3	D
G.	5	C
H.	10	E, F, G
I.	none	H

The company has enough money to hire one extra person to help with these project.

Option 1: Hire someone to help with task C, which would reduce the time it takes to complete task C by 5 days

Option 2: Hire someone to help with task E, which would reduce the time it takes to complete task E by 4 days

Given the company wants to complete this project in as little time as possible, should they choose option 1 or option 2, and how much time will be save?

3.1 EXPONENTS REVIEW

An exponent is a number (or variable) that tells us how many times to multiply the base by itself.

exponent

$y = a^x$

base

$3^5 = 3 \times 3 \times 3 \times 3 \times 3$

$x^5 = x \times x \times x \times x \times x$

Ex: Calculate each of the following expressions.

$7^3 =$

$(0.25)^8 =$

$8(1.025)^3 =$

$1000(0.95)^7 =$

Practice Questions

1) Calculate each of the following expressions.

a) $12^3 =$

b) $(0.87)^{12} =$

c) $500(1.6)^2 =$

d) $2000(0.82)^{15} =$

3.2 POPULATION AND COMPOUND INTEREST

We can use an exponential function for questions about population or compound interest.

$$C_n = a(rate)^n$$

Where: a is the initial value (starting amount)

C_n is the final value (ending amount)

rate is how fast something is increasing or decreasing

n is the time

*We always have to make sure that time is measured in the same units as the rate of growth/decay.

We can determine rate as follows:

If we are given words: “doubles”, $c = 2$. “triples”, $c = 3$. “half”, $c = 1/2$ (or 0.5), etc.

If we are given a percent:

If increase: $rate = 1 + \frac{\%}{100}$

If decrease: $rate = 1 - \frac{\%}{100}$

Ex: An initial population of 1000 bacteria increases at a rate of 3% per day. What is the population after 4 days?

Ex: An initial population of 800 fish decreases by half every 6 months. What will the population be in 4 years?

Population and Financial Math Unit – Population and Compound Interest

Ex: A city has a population of 30 000 people. The population is declining at a rate of 12% per year. What will the population be after 7 years?

Ex: Miguel invests $500 at an annual compound interest rate of 2.6%. How much is Miguel's investment worth after 8 years?

Ex: You borrow $700 at a monthly compound interest rate of 1.5%. How much will you have to repay after 2 years?

Practice Questions

1) 140 students attend an art camp. It is expected that this number will increase at a rate of 4% every year. What will the population be after 5 years?

2) In an aquatic environment, there are 480 different species. It is expected that the number of species will decrease by 10% every year. How many species will there be after 8 years?

3) There are 4 bacteria in a petri dish. The population doubles every 20 minutes. What will the population be after 4 hours?

4) \$5000 is invested at an interest rate of 6% compounded annually. What is the value of this investment in 5 years?

5) You purchase a new computer for \$1200. The value of the computer depreciates at a rate of 43% annually. How much will your computer be worth after 3 years?

6) Valerie invests \$2500 at a quarterly compound interest rate of 2.3%. What is the value of her investment after 6 years?

3.3 SOLVING FOR OTHER VARIABLES

If we need to solve for either a or c, we can plug in the variables we know and use algebra to solve, or we can use the following formulas.

$$a = \frac{c_n}{rate^n} \quad \text{and} \quad rate = \sqrt[n]{\frac{c_n}{a}}$$

Note: When solving for percent, remember $rate = 1 \pm \frac{\%}{100}$, so you need to do a few more calculations to determine the percent.

Ex: It is expected that the value of a stock will triple every 6 months. After 1 year, the value of the stock is $10.80. What was the initial value of the stock?

Ex: A grocery basket costs $240 in 2018. If the cost of the same basket will be $269.97 in 2021, what was the annual inflation rate during this time?

Practice Questions

1) A student posts a photograph online. On the first day the photo is posted, it is seen by 38 people. After 10 days, the photo has been seen by 2191. What is the rate by which the number of people seeing the photo increases each day?

2) Calcium chloride is used to de-ice roads during Quebec's winters. During an ice storm, 7.5 cm of ice accumulates on the roads because the de-icer spreads the salt. After 3 hours the thickness of the ice will be 5.11 cm? By what rate does the salt reduce the thickness of ice per hour?

3) Super Balls are made out of a synthetic rubber. They bounce much higher than other balls. At each bounce, a Super Ball loses 8% of its maximum height from the previous bounce. After 8 bounces, the ball has a height of 200 cm. From what height was the ball originally dropped?

4) After 8 years, the accumulated capital of an investment at an annual compound interest rate of 2% will be $7029.96. What was the amount of the initial investment?

5) Money is invested at an interest rate of 4% compounded annually. If the future value of this investment after 6 years is \$3796, what is the value of the initial investment?

6) \$4700 was invested and yields \$8699.37 after 8 years. What was the annual compound interest rate?

7) \$2800 was invested and after 2 years the value of the investment was \$3087. What was the annual compound interest rate?

8) At what annual compound interest rate should we invest so that our initial investment will be doubled after 10 years?

3.4 LOGARITHM BASICS

We have solved exponential functions for every variable except x. In order to solve for x, we need to use logarithms.

A logarithm is a function (and a button on your calculator, just like the trig functions: sine, cosine, and tangent).

We can re-write an exponential function as a logarithmic function.

$$y = c^x \Leftrightarrow x = \log_c y$$

Ex: Write the following expressing using logs

a) $4 = 5^x$

b) $150 = 3^x$

c) $6561 = 3^8$

To solve for x, we can use the following formula

$$x = \frac{\log of\ the\ lonely\ number}{\log of\ the\ number\ with\ the\ exponent}$$

Ex: Solve for x

$8 = 2^x$

$150 = 4(1.05)^x$

Practice Questions

1) Solve each of the following for x.

a) $20 = 3^x$

b) $1.79 = 2.5^x$

c) $120.5 = 1.03^x$

3.5 USING LOGS TO SOLVE POPULATION AND FINANCIAL QUESTIONS

In order to solve an exponential function for x you can either plug in what you know and use algebra (remembering to solve using logs) or you can use the following formula:

$$n = \frac{\log\left(\frac{C_n}{a}\right)}{\log(rate)}$$

Ex: Bill has 2g of cells in a petri dish. The amount of cells increases by 12% every month. How long does it take for Bill to have 4g of cells?

Ex: An initial investment of $3000 earns an annual compound interest rate of 2%. How long will it take for the investment to be worth $3312.24?

Practice Questions

1) $2500 is invested at an annual compound interest rate of 3%. How long will it take for the investment to be worth $2985?

2) $5000 is invested at an annual interest rate of 2% compounded annually. How long will it take for the investment to be worth $5520.40?

3) A petri dish contains 150 bacteria. The number of bacteria increases by 20% every hour. How long will it take for there to be 500 bacteria?

4) Water from a tank evaporates at a rate of 1% of its volume every hour. After how many hours will the tank hold 4184 L of water, if the tank initially contained 4650 L of water?

Population and Financial Math Unit – Simple Interest
3.6 SIMPLE INTEREST

When working with financial math (ex: finding the future value of an investment, etc.), the question may ask about simple interest or compound interest.

Simple interest means you only earn interest on your initial investment. For example, if you invest $500 at an annual interest rate of 1%, you would earn 1% of $500 (or $5) for each year you kept your money in the investment.

We do not need exponential functions or logs to calculate investments with simple interest, but instead use the formula:

$$C_n = a(1 + rate * n)$$

Where:

C_n is the final amount

a is the initial investment

rate is the interest rate $\left(\frac{\%}{100}\right)$

n is the time (if necessary, transform t so it is in the same units as the interest rate)

Ex: Jill invests $5000 at a simple interest rate of 6% annually. How much money does Jill have after 5 years?

Ex: Ryan invests $1000 at an annual simple interest rate of 4%. How much money does Ryan have after 18 months?

Practice Questions

1) Determine the accumulated capital (the final amount of the investment) if you invested $8000 for 5 years at an annual simple interest rate of 7%.

2) Carlos invests $3600 at a monthly simple interest rate of 0.45%. What is the value of Carlos' investment after 7 years?

3) Mikka is saving money to buy a car. She wants to have at least $5000 for the down payment and so she makes two investments.

Investment 1: Mikka invests $2247.19 at a monthly simple interest rate of 1.3% for 3 years.

Investment 2: Mikka invests $1200 at a weekly simple interest rate of 0.3% for 3 years.

After 3 years, will Mikka have reached her goal?

3.7 SOLVING FOR OTHER VARIABLES

We will not always be asked to solve for the accumulated capital (final value) of an investment. We can solve for the other variables by plugging in what we know and using algebra to solve, or by using the following formulas:

$$C_0 = \frac{C(t)}{1+it} \qquad t = \frac{\frac{C(t)}{C_0} - 1}{i} \qquad i = \frac{\frac{C(t)}{C_0} - 1}{t}$$

Ex: Rasha invested some money for 11 years at an annual simple interest rate of 7.4%. After 11 years, the investment was work $7890.90. How much did Rasha invest initially?

Ex: How long will it take an initial capital of $9300 to earn an accumulated capital of $14 322 at an annual simple interest rate of 8%?

Ex: What is the annual simple interest rate for an investment of $2400 that yields $4646.40 after 9 years.

Practice Questions

1) After 7.5 years, an investment is worth \$4199.75. What was the amount of the initial investment given that the simple interest rate was 3.8% per half-year?

2) \$1800 is invested at a monthly simple interest rate of 2.25%. How long does it take for the investment to be worth \$3420?

3) \$9000 is invested for 7 years and 4 months. It is now worth \$18 504. What was the monthly simple interest rate?

4) You borrowed \$3700 at a daily simple interest rate of 0.02%. When you repaid the loan, you paid \$4780.40. How long was it before you repaid the loan?

5) After 2.5 years, the repayment of a debt will be \$11 770.75 at a weekly simple interest rate of 0.15%. How much money was borrowed initially?

6) The repayment of a \$7 350 loan is \$14 891.10 after 9 years. Determine the quarterly simple interest rate.

3.8 POPULATION AND FINANCIAL MATH EXAM STYLE QUESTIONS

Multiple Choice

1) Keana initially invested \$5000. She would like to know exactly when her investment will be doubled. She uses the expression below.

$$C_n = 5000(1.03)^n$$

where C_n is the future value and *n* is the number of years.

Which of the following equations would be used to calculate the exact amount of time it will take for her investment to double?

A) $n = \frac{\log 1.03}{\log 2}$

B) $n = \frac{\log 10\ 000}{\log 5000}$

C) $n = \frac{\log 10\ 000}{\log 2}$

D) $n = \frac{\log 2}{\log 1.03}$

Short Answer

2) Tarek invested an amount of money 5 years ago at an interest rate of 3% compounded every six months.

The value of Tarek's investment can be determined by using the following rule.

$$C_n = a(1.03)^n$$

where *n*: number of 6-month periods elapsed since the beginning of the investment

C_n: value of the future investment, in dollars (\$)

Today, the value of Tarek's investment is \$1612.70.

To the nearest dollar, what was the value of Tarek's initial investment?

Long Answer

3) For the new Financial Math course, students have started a virtual investment club. The club members have been divided into three teams.

- Team A and Team B calculated their projected investment after 4 years.
- Team A initially invested $2000 with an annual compound interest rate of 3.4%.
- Team B initially invested $1500 with an interest rate of 12% compounded every 6 months.

Team C invested $3500 with an annual compound interest rate of 7.5%. It wants the value of its investment to equal the sum of Team A's and Team B's projected 4-year investment.

How long will it take for the value of Team C's investment to be equal to the sum of Team A's and Team B's projected investment?

4) Marianna is interested in buying a used car. During her visit to the dealership, she is hesitating between two cars. Both cars are 5 years old.

- The first car was originally priced at $18 000 and has depreciated at a rate of 15% annually.
- The second car was originally priced at $20 000 and has depreciated at a rate of 9% every 6 months.

A number of years ago, Marianna invested $6000 at a rate of 4.5% compounded annually.

She now has exactly enough money to buy the cheaper of the two cars.

How many years ago did Marianna invest the money?

Voting Unit – Plurality
4.1 PLURALITY

There are many different ways to count votes in order to determine a winner.

- Canada, the United States, and many countries in Africa use **plurality**.
- The Australian House of Representative, the Indian presidential election, and parliament in Papau New Guinea use **majority with elimination**.
- Nauru, a tiny island country in Micronesia (just northeast of Australia) uses **Borda count**.
- The **Condorcet method** is not practical to use in elections, but a modified version of it is the basis for voting in Robert's Rules of Order.
- Most countries in western Europe and South America, as well as Russia, Kazakhstan, and several countries in Africa use **proportional representation**.

Each method has advantages and disadvantages.

Plurality

In voting under a plurality method, people vote for their favorite option. The option with the most votes is the winner.

Advantages: it is easy to calculate and a winner is guaranteed (unless there is a tie)

Disadvantages: A winner could emerge that most people do not want

Ex: Grade 11 students are electing their class president. Of the 150 students in grade 11, 70 vote for Ali, 30 vote for Brenda, and 50 vote for Chris. Under plurality, who wins the election?

In some questions, we will be given a ranked ballot where people put all the options in order.

Ex: Dr. James is going to bring a snack (apples, granola bars, or pepperoni sticks) in for her math students and the students will vote to determine the snack. The following table summarizes the results of the student vote.

# of votes / Preference	45	32	28	23
1st choice	apples	granola bars	pepperoni	granola bars
2nd choice	granola bars	apples	apples	pepperoni
3rd choice	pepperoni	pepperoni	granola bars	apples

Given these results, which snack would win under plurality?

Practice Questions

1) In a hockey league, a committee of 53 members must select a recipient for the trophy for the hardest-working player from a list of three candidates.

- Candidate A received 16 votes
- Candidate B received 20 votes
- Candidate C received 17 votes

Under plurality voting, which candidate wins?

2) To determine Monday's menu at a high school cafeteria, the school's 250 grade 11 students were asked to rank three menus in order of preference. The results are presented in the table below.

# of votes / Preference	63	51	46	45	24	21
1^{st} choice	Pizza	Hamburgers	Tacos	Tacos	Hamburgers	Pizza
2^{nd} choice	Hamburgers	Pizza	Pizza	Hamburgers	Tacos	Tacos
3^{rd} choice	Tacos	Tacos	Hamburgers	Pizza	Pizza	Hamburgers

Under plurality voting, which menu item would be served?

3) A new school is being built and an election was held to determine if it will be in Village A, Village B, or Village C. Village A received 45% of the vote. Village B received 35% of the vote. Village C received 20% of the vote. Under plurality rule, where will the school be built?

Voting Unit - Majority

4.2 MAJORITY

In voting under a majority method, people vote for their favorite option. The option with more than half the votes is the winner.

Step 1: determine the total number of votes

Step 2: divide the total number of votes by 2

Step 3: determine the number of first place votes for each option

Step 4: determine a winner (the option with **more than half** the votes).

** Note: if no option has more than half the votes, there is no winner.*

Advantages: more than half the people are guaranteed to like the winner

Disadvantages: there may be no winner

Ex: A group of friends is trying to decide where to eat dinner. They decide to vote for their favorite type of food, and the results are presented in the table below. Which option wins using majority?

# of votes / Preference	4	4	7	5
1^{st} choice	Vietnamese	Italian	Vietnamese	Sushi
2^{nd} choice	Sushi	Vietnamese	Italian	Italian
3^{rd} choice	Italian	Sushi	Sushi	Vietnamese

Ex: A group of people are voting on their favorite winter activity. 70 vote for skiing, 30 vote for snowshoeing, and 50 vote for hockey. Under majority, which option wins?

Practice Questions

1) The 40 members of the board of directors of a company must vote to elect the president of the board. Three candidates apply for the position and the results of the vote are presented in the table below.

# of votes / Preference	16	14	10
1st choice	A	C	B
2nd choice	B	A	A
3rd choice	C	B	C

Given the board of directors use majority to determine the winner, which candidate wins?

2) Grade 11 students are voting on where to go on their class trip. The results are presented in the table below.

# of votes / Preference	63	51	46	45	24	21
1st choice	Montreal	Montreal	Vancouver	Vancouver	Vancouver	Montreal
2nd choice	Halifax	Halifax	Montreal	Halifax	Halifax	Vancouver
3rd choice	Vancouver	Vancouver	Halifax	Montreal	Montreal	Halifax

Under majority voting, which location would win?

4.3 MAJORITY WITH ELIMINATION

In voting under a **majority with elimination** method, people rank the options from favorite to least favorite. The option with more than half the votes is the winner. However, if no option receives more than half the votes, the option with the fewest votes is eliminated and re-assigned.

Step 1: determine the total number of votes

Step 2: divide the total number of votes by 2

Step 3: determine the number of first place votes for each option

Step 4: determine a winner (the option with **more than half** the votes)

Step 5: If no winner, the option with the fewest number of votes is eliminated and the votes are re-assigned to the next option on the list(s). Repeat this step until an option has more than half the votes

Advantages: A winner is guaranteed and people's preferences for all options are included

Disadvantages: Voting and counting the votes can be time-consuming depending on the number of options

Ex: A political party is holding elections to determine its new leader. There are 5 people nominated for the position.

# of votes / Preference	23	56	60	20	41
1st choice	A	B	C	A	D
2nd choice	B	A	B	E	B
3rd choice	C	C	A	C	A
4th choice	D	D	E	D	C
5th choice	E	E	D	B	E

Ex: A group of 150 people were asked to rank their favorite snowmobile brands. The results are in the table below.

# of votes / Preference	26	21	30	19	54
1st choice	Polaris	Yamaha	Ski-Doo	Polaris	Yamaha
2nd choice	Ski-Doo	Polaris	Arctic Cat	Arctic Cat	Polaris
3rd choice	Yamaha	Arctic Cat	Polaris	Yamaha	Ski-Doo
4th choice	Arctic Cat	Ski-Doo	Yamaha	Ski-Doo	Arctic Cat

Voting Unit – Majority with Elimination

Practice Questions

1) The table below presents the results of an election between 3 candidates: A, B, and C.

# of votes / Preference	50	30	27	24	10
1st choice	C	B	A	A	B
2nd choice	B	C	B	C	A
3rd choice	A	A	C	B	C

Which candidate wins using majority with elimination?

2) Philemon Wright is selling grad hoodies, but can only order 1 color of hoodies. Students are asked to vote among the following options: Green, Blue, Red, and Black. The results of the vote are presented in the table below:

# of votes / Preference	52	41	35	33	29	21
1st choice	Blue	Green	Green	Black	Blue	Red
2nd choice	Green	Blue	Black	Blue	Green	Green
3rd choice	Black	Red	Red	Red	Red	Black
4th choice	Red	Black	Blue	Green	Black	Blue

Using majority with elimination, which color hoodie will be ordered?

Voting Unit – Borda Count

4.4 BORDA COUNT

In voting under a **Borda count** method, people rank the options from favorite to least favorite. Points are assigned to each choice. For each option, the number of votes is multiplied by the points, and all are added together. The option with the most points is the winner.

Step 1: Assign points to each option (last choice is 0, next is 1, next is 2, and so on until all choices are assigned)

Step 2: Every time an option is ranked, the number of votes is multiplied by the points assigned

Step 3: All the points for each option are added together

Step 4: The option with the most votes is the winner.

Advantages: This allows for a nuanced interpretation of preferences and will generally lead to a high degree of satisfaction among the electorate

Disadvantages: It's a complicated system to implement

Ex: A group of people are asked about what kind of movies they like best. They are given 3 options to rank: comedy, horror, and documentary. The results are shown in the table below.

# of votes / Preference	45	32	28	23
1st choice	Comedy	Documentary	Horror	Documentary
2nd choice	Documentary	Comedy	Comedy	Horror
3rd choice	Horror	Horror	Documentary	Comedy

Given these results, which move type is most popular using Borda count?

Ex: A town is electing a new mayor. There are 4 candidates (A, B, C, and D). The results are presented in the table below.

# of votes / Preference	300	450	430	380
1st choice	A	C	D	D
2nd choice	C	B	B	B
3rd choice	B	A	C	A
4th choice	D	C	A	C

Practice Questions

1) A group of friends is trying to decide how to spend a Saturday. They take a vote and the results are presented in the table below. Which option wins using Borda count?

# of votes / Preference	4	4	7	5
1st choice	Movie	Video Games	Movie	Museum
2nd choice	Museum	Movie	Video Games	Video Games
3rd choice	Video Games	Museum	Museum	Movie

2) Grade 11 students were asked to rank the following 3 classes from favorite to least favorite: Math, English, and French. The results are presented in the table below. Which option wins using Borda count?

# of votes / Preference	63	51	46	45	24	21
1st choice	Math	Math	English	English	French	Math
2nd choice	English	French	Math	Math	English	French
3rd choice	French	English	French	French	Math	English

4.5 CONDORCET METHOD

The Condorcet method is a voting system in which the winner is the option that, when compared in a head-to-head competition with every other method, is the preferred option.

Step 1: Compare option A to option B. Determine the number of people who prefer A to B and the number of people who prefer B to A. The option with the most votes is the winner.

Step 2: Compare option A to another option and repeat step 1. Do this for as many options as exist.

Step 3: Compare option B to every other option, one option at a time (as in step 1).

Step 4: Repeat with every possible combination of options.

Step 5: Declare the winner – the option that never loses.

Advantages: this method is the most rigorous and assures the most number of people will be satisfied by the outcome.

Disadvantages: there is often no winner and it is complicated to implement.

Ex: The 40 members of the board of directors of a company must vote to elect the president of the board. Three candidates apply for the position and the results of the vote are presented in the table below.

# of votes / Preference	16	14	10
1st choice	A	C	B
2nd choice	B	A	A
3rd choice	C	B	C

Given the board of directors use the Condorcet method to determine the winner, which candidate wins?

Voting Unit – Condorcet Method

Ex: Voters are asked to choose between 4 candidates: A, B, C, and D. The results of the vote are presented in the table below:

# of votes / Preference	70	60	50	45
1st choice	A	D	C	B
2nd choice	B	A	D	C
3rd choice	C	B	A	D
4th choice	D	C	B	A

Using the Condorcet method, which candidate wins?

Practice Questions

1) Philemon Wright is selling grad hoodies, but can only order 1 color of hoodies. Students are asked to vote among the following options: Green, Blue, Red, and Black. The results of the vote are presented in the table below:

# of votes / Preference	52	41	35	33	29	21
1st choice	Blue	Green	Green	Black	Blue	Red
2nd choice	Green	Blue	Black	Blue	Green	Green
3rd choice	Black	Red	Red	Red	Red	Black
4th choice	Red	Black	Blue	Green	Black	Blue

Using the Condorcet method, which color hoodie will be ordered?

2) A small group of people are asked their favorite fast-food restaurant. The results are presented in the table below.

# of votes / Preference	4	6	5
1st choice	McDonalds	Taco Bell	Pizza Pizza
2nd choice	Taco Bell	Pizza Pizza	McDonalds
3rd choice	Pizza Pizza	McDonalds	Taco Bell

Using the Condorcet method, which option is the most preferred?

Voting Unit – Proportional Representation
4.6 PROPORTIONAL REPRESENTATION

In **proportional representation**, individuals vote for the party they most prefer. The total number of votes for each party are calculated, and then the parties are awarded a number of seats in proportion to the number of votes received.

Step 1: Determine the number of votes for each party

Step 2: Determine the total number of votes

Step 3: Determine the proportion of votes for each party $\frac{votes\ for\ the\ party}{total\ number\ of\ votes}$

Step 4: Multiply the proportion of votes for each party by the number of seats available

Step 5: Ignore the decimals (do not round) and assign each party that number of seats

Step 6: Assign any remaining seats by choosing the highest decimal, the next highest, etc until all seats are assigned

Advantages: This system is quite representative of individual preferences and leads to a distribution of power that fairly accurately reflects the will of the electorate

Disadvantages: There is often no majority parts and thus requires a coalition government that can slow the decision making process.

Ex: A country is electing representatives to fill 90 seats in its Parliament. There are 5 parties in the election: A, B, C, D, E. The results of the election are presented in the table below.

Party	Number of Votes
Party A	113
Party B	108
Party C	132
Party D	86
Party E	92

Using proportional representation, how many seats will each party earn?

Voting Unit – Proportional Representation

Ex: A town is holding an election to fill 12 seats on its council. There are 3 parties and the results of the election are in the table below:

Party	Percent of Vote Received
A	38%
B	33.6%
C	28.4%

Using proportional representation, how will the available seats be divided?

Practice Questions

1) In 2019, Canada held a federal election and the number of popular votes each party received is in the table below.

Party	# of Votes
Liberal	6 018 728
Conservative	6 239 227
Bloc Quebecois	1 387 030
New Democratic Party	2 903 722
Green Party	1 189 607
Independent	72 546

a) If Canada used a proportional representation system, how many of the 338 seats in Parliament would each party receive?

b) The actual number of seats won by each party under our current system (districts using plurality) is in the table below. How would proportional representation change these results?

Party	# of Seats
Liberal	157
Conservative	121
Bloc Quebecois	32
New Democratic Party	24
Green Party	3
Independent	1

Voting Unit – Approval

4.7 APPROVAL

The final method of voting we will consider is Approval voting.

In approval voting, instead of ranking candidates, voters will select as many options as they like. Every option they choose will receive 1 vote. The option with the most votes is the winner.

Ex: The results of an election are presented in the table below.

Number of Voters	45	32	28	23
	A D	B C D	A B C	A

Using approval voting, which option would win?

Practice Questions

1) A group of friends is trying to decide where to eat dinner. They decide to vote for their favorite type of food, and the results are presented in the table below. Which option wins using Approval voting?

# of votes	4	4	7	5
	Vietnamese Sushi	Italian Vietnamese	Vietnamese Italian Sushi	Sushi

2) To determine Monday's menu at a high school cafeteria, the school's 250 grade 11 students were asked to rank three menus in order of preference. The results are presented in the table below.

# of votes	63	51	46	45	24	21
	Pizza Hamburgers	Pizza Tacos	Tacos	Tacos Hamburgers	Hamburgers	Pizza

4.8 EXAM STYLE QUESTIONS

Long Answer

1) The Grade 11 class at your high school is planning the graduation trip. Three possible destinations are offered to the students: New York, Boston and Chicago. The three trips are all comparable in price. All 241 grade 11 students were asked to give their order of preferences as to which destination they would like to visit. The results were recorded in the table below.

Ranking \ # Votes	86	40	30	85
1st	New York	Boston	Boston	Chicago
2nd	Boston	Chicago	New York	Boston
3rd	Chicago	New York	Chicago	New York

The school administration agreed to choose the winning destination after analysing the results of the four following voting procedures: Borda, Elimination, Plurality and Condorcet**.**

Which destination was chosen by the school administration?

Voting Unit – Exam Style Questions

2) The grad committee will be selling hoodies to the 240 graduating students.

Three colours have been proposed for the hoodies: black, red, and green.

The grad committee asked the students to rank these colours in order of preference. The results of the vote are shown in the table below.

NUMBER OF VOTES	**$\frac{1}{3}$ of the students**	**10 % of the students**	**?**	**15% of the students**
1ST CHOICE	black	red	green	black
2ND CHOICE	red	green	red	green
3RD CHOICE	green	black	black	red

After analyzing the results of the vote, the students were told that the chosen colour was green.

Pamela, Erin, Brandon and Charlie made the following affirmations.

- Pamela thinks that the colour was chosen using plurality voting.
- Erin thinks that the colour was chosen using the elimination method.
- Brandon thinks that the colour was chosen using Borda count.
- Charlie thinks that the colour was chosen using the Condorcet method.

Among Pamela, Erin, Brandon and Charlie, whose affirmation is correct? Select all that apply.

Voting Unit – Exam Style Questions

3) Students were asked to rank 3 course options according to their preferences.

All 930 students in the school answered the survey. The results are indicated in the table below.

Number of students who ranked the course options in this way	**20% of student population** **________ students**	**$\frac{1}{3}$ of the student population** **________ students**	**434 students**
1st choice	Psychology	Biology	Media Studies
2nd choice	Biology	Media Studies	Biology
3rd choice	Media Studies	Psychology	Psychology

The teachers analyzed the data collected from the students using: Plurality voting, Borda count, Elimination method and Condorcet method.

The teachers made the following prediction: in three out of the four methods, the same course option will be ranked as their first choice.

Are the teachers correct in their prediction? Justify each of the methods.

Probability Unit – Review
5.1 REVIEW

Recall that there are several types of probability: theoretical, experimental, and subjective.

- The **theoretical probability** of an event is a number that quantifies the possibility that the event will occur – it is determined by mathematical reasoning.

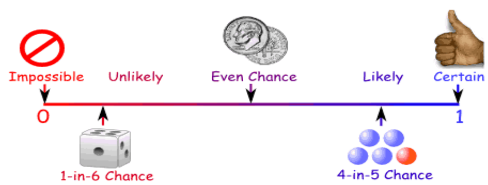
$$Theoretical\ Probability = \frac{\#\ of\ favorable\ outcomes}{\#\ of\ possible\ outcomes}$$

- The **experimental probability** of an event is determined from the experiment itself.

$$Experimental\ Probability = \frac{\#\ of\ times\ outcomes\ occurs}{\#\ of\ trials}$$

- The **subjective probability** that an event will occur relies on judgment or perceptiveness of a person who has a particular set of information about the situation.

Probability is the chance something will happen. This can be shown on a probability line.

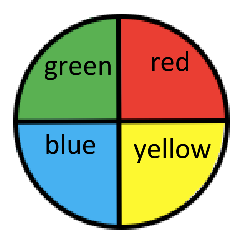


We can also use numbers (such as fractions or decimals) to show the probability of something happening.

- Impossible is a zero
- Certain is a one

We can calculate the probability of an event using the following formula:

Probability of Event A: P(A)
$P(A) = \dfrac{Number\ of\ ways\ A\ occurs}{Total\ number\ of\ ways}$



Ex: A spinner has 4 equal sectors coloured yellow, blue, green and red.

What are the chances of landing on blue?

What are the chances of landing on red?

Ex: Where would the following statements sit on the probability line?

A) Choosing a green ball from a bag with only 4 green balls

B) Choosing a green ball from a bag with only 2 red balls and 4 yellow balls

C) Choosing a green ball from a bag with 2 green balls and 2 red balls

D) Choosing a red ball from a bag with 1 red ball and 3 green balls

Probability Unit – Review

Practice Questions

1) A jar contains 100 jelly beans. There are 20 red jelly beans, 10 black jelly beans, 15 orange jelly beans, 30 green jelly beans, and 25 yellow jelly beans. If you chose a single jelly bean at random, what is the probability that you would choose an orange jelly bean?

2) In a standard deck of 52 playing cards, what is the probability of randomly drawing:

a) a queen?

b) a red card?

c) a club?

d) the 7 of diamonds?

3) A bag contains 4 red marbles, 2 green marbles, 6 white marbles, and 3 blue marbles. What is the probability of randomly drawing:

a) a red marble?

b) a green marble?

c) a white marble?

d) a blue marble?

Probability Unit – Probabilities with “And” or “Or”

5.2 PROBABILITIES WITH “AND” OR “OR”

The probability of A and B means we want to know the probability of two events happening at the same time. For now, we will only consider **independent events**. This means the probability of one event does not change the probability of another event.

Consider the following: you flip a coin and roll a die. The result of the coin flip does not alter the roll of the die – these are **independent events**.

Now consider drawing two cards from a deck of 52 cards. After you draw the first card, there are fewer cards to draw for the second card, thus changing the probabilities. These are **dependent events**, and we will address them later.

To determine the probability of two independent events occurring, multiply probabilities:

$$P(A \cap B) = P(A) \times P(B)$$

Ex: Dr. James draws one card from a deck of 52 cards and flips a coin once. What is the probability she drew a 5 and the coin landed on tails?

Ex: You roll two dice. What is the probability you roll a 6 on both?

Ex: You roll two dice. What is the probability you roll a 4 on one die and a 2 on the other?

Probability Unit – Probabilities with "And" or "Or"

The probability of A or B means we want to know the probability of either of the two events happening.
We will continue to consider only independent events.

To determine the probability either of two independent events occurring:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex: Dr. James draws one card from a deck of 52 cards and flips a coin once. What is the probability she either drew a 5 or the coin landed on tails?

Ex: You roll a green die and a blue die. What is the probability you roll a 3 on the green die or a 5 on the blue die?

Ex: You roll two dice. What is the probability you roll a 4 on one die and a 2 on the other?

Practice Questions

1) You draw one card from a standard deck of cards. What is the probability you draw a 4 or a queen?

2) You draw one card from a standard deck of cards. What is the probability you draw a 7 or a red card?

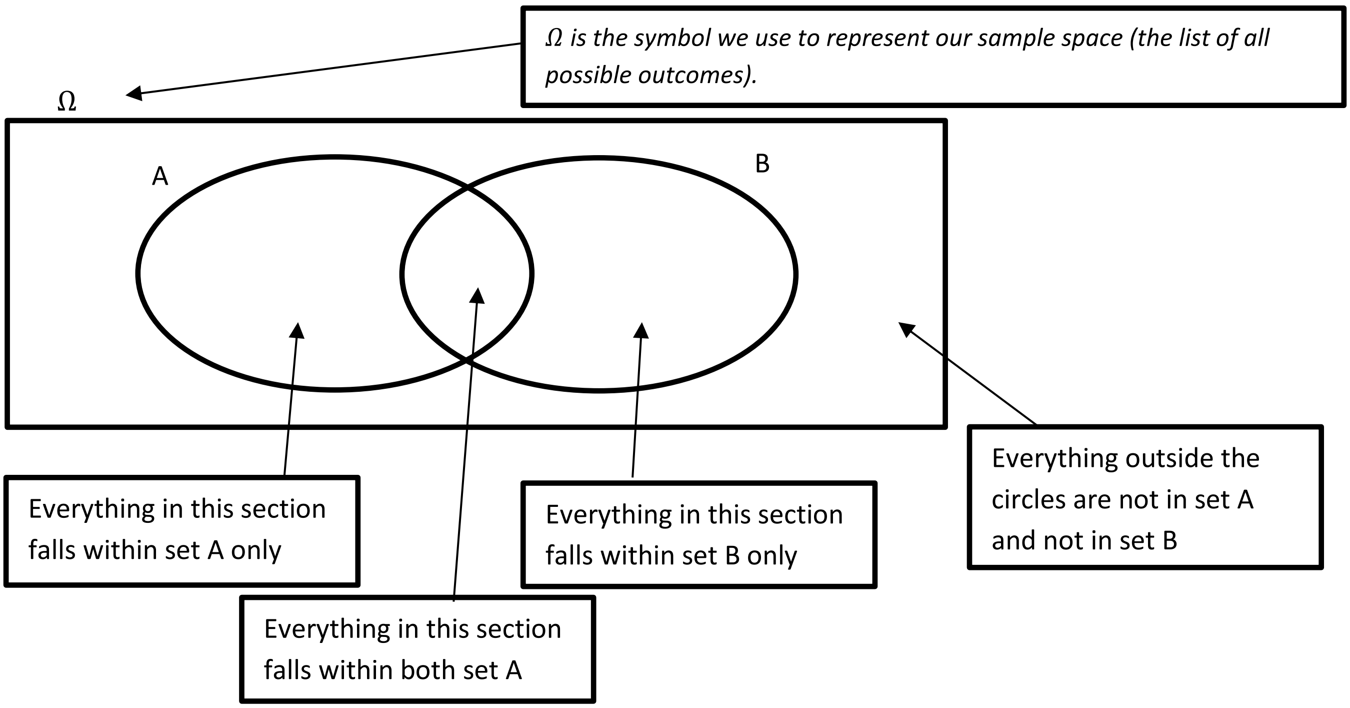
3) You roll two dice. What is the probability you roll a 4 and a 6?

4) You roll two dice. What is the probability you roll a 4 or a 6?

5) You draw a marble from a bag containing 4 blue marbles and 3 red marbles. You also flip a coin. What is the probability you draw a blue marble and the coin lands on heads?

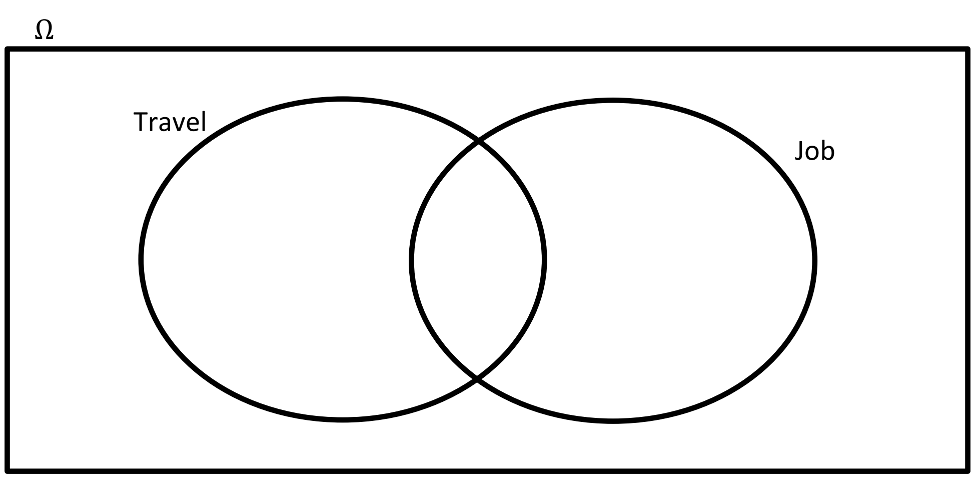
5.3 VENN DIAGRAMS

Venn diagrams are great because we can use them to think about probability for independent of dependent events. We won't even have to think about whether the events are independent or dependent.



Ex: Dr. James asked 30 students the following questions: 1) Do you travel more than 30 minutes to get to school? 2) Do you have a job? Use the information in the table below to create a Venn diagram.

Yes to Q1 only (travel)	**Yes to Q2 only (job)**	**Yes to both Q1 and Q2**	**No to both**
8 students	6 students	7 students	9 students

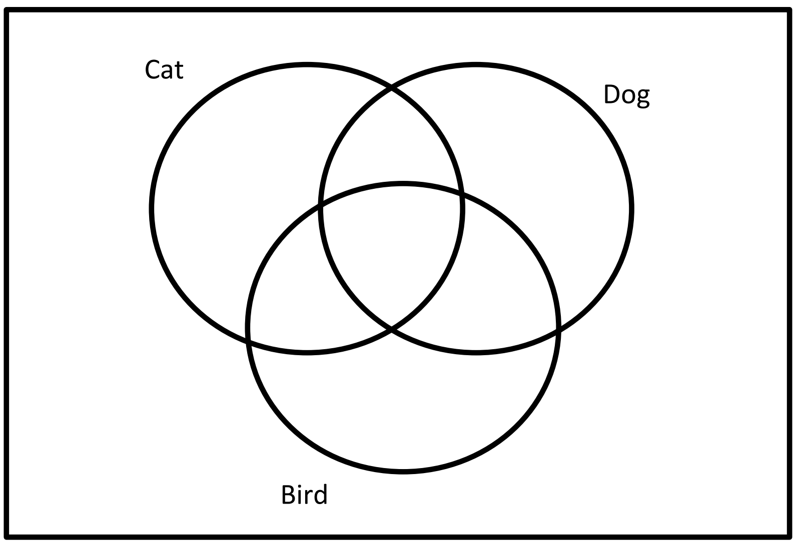


Probability Unit – Venn Diagrams

We can also create Venn Diagrams with more than two circles.

Ex: Dr. James asked 30 students the following questions: 1) Do you own a cat? 2) Do you own a dog? 3) Do you own a bird? The results of the survey are presented in the table below and depicted in the Venn Diagram. Use the information in the table below to create a Venn diagram.

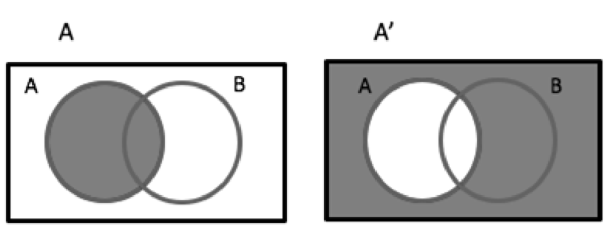
Yes to Cat only	Yes to Dog only	Yes to Bird only	Yes to Cat and Dog	Yes to Cat and Bird	Yes to Dog and Bird	Yes to Cat, Bird, and Dog	No to all
4 students	5 students	3 students	6 students	1 student	2 students	0	9 students



Venn diagrams are especially useful for determining "AND" and "OR" probabilities. They also let us visualize the complement of an event. The **complement** of an event is the collection of outcomes that are NOT the event.
In the spinner example, if the event (A) is spinning the spinner and landing on BLUE, the complementary event (A') is landing on NOT BLUE (everything other than blue).

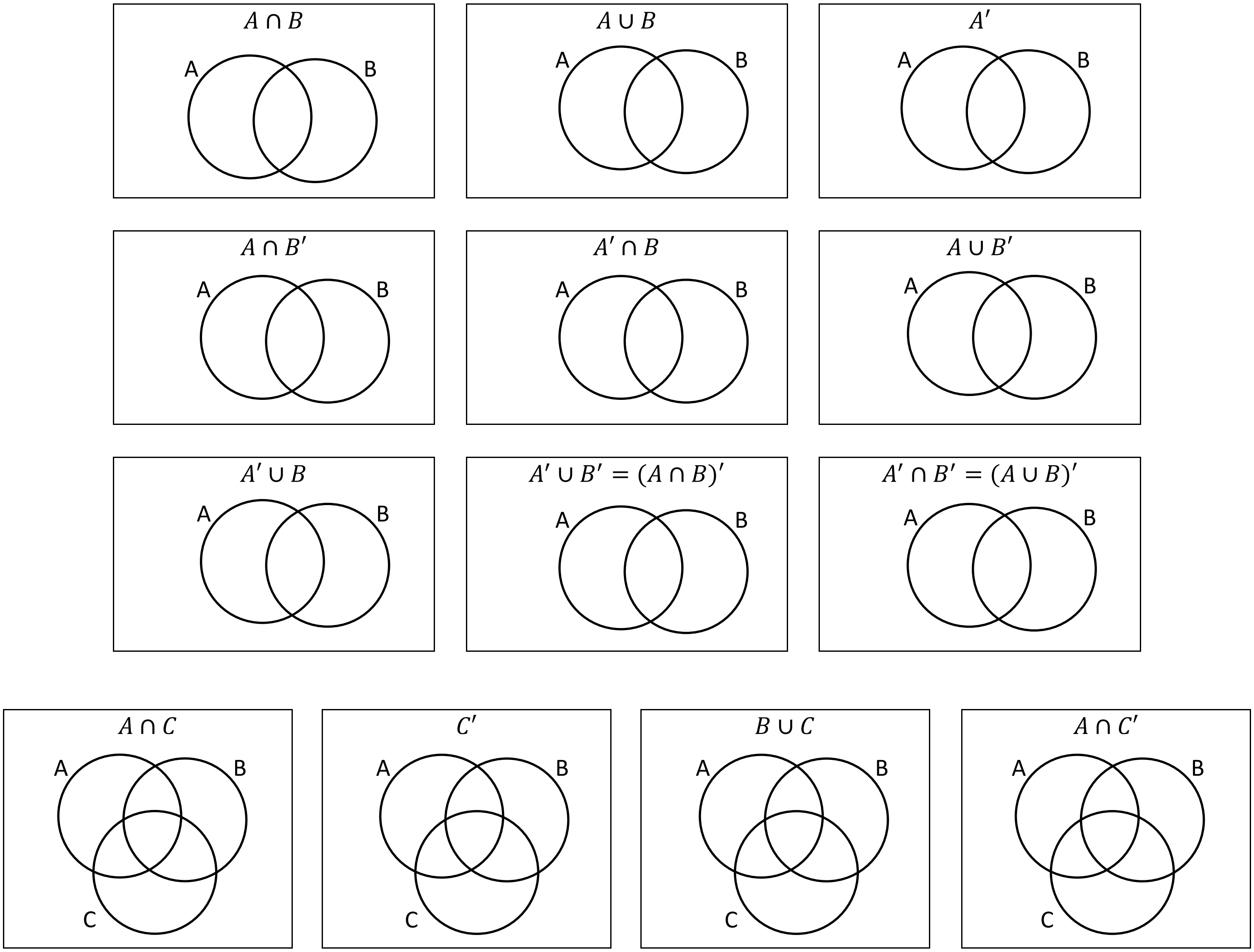
When you add the probability of an event, (A) to the probability of the complement of the event (A'), the result will always be 1. It is sometimes easier to determine P(A) using this method:

$$P(A) + P(A') = 1$$



Probability Unit – Venn Diagrams

Ex: Shade the appropriate section of each Venn diagram.

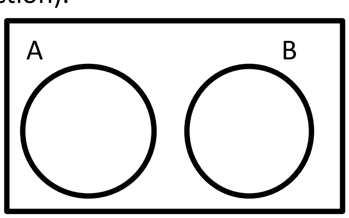


Events are called **mutually exclusive** if they cannot happen at the same time.

Ex: If we are spinning a spinner one time and Event A=spinning BLUE and Event B=spinning RED, those events are mutually exclusive because you cannot land on BLUE and RED at the same time.

In a Venn Diagram, two mutually exclusive events do not overlap (or if the circles are drawn as overlapping, there is nothing in that overlapping section).

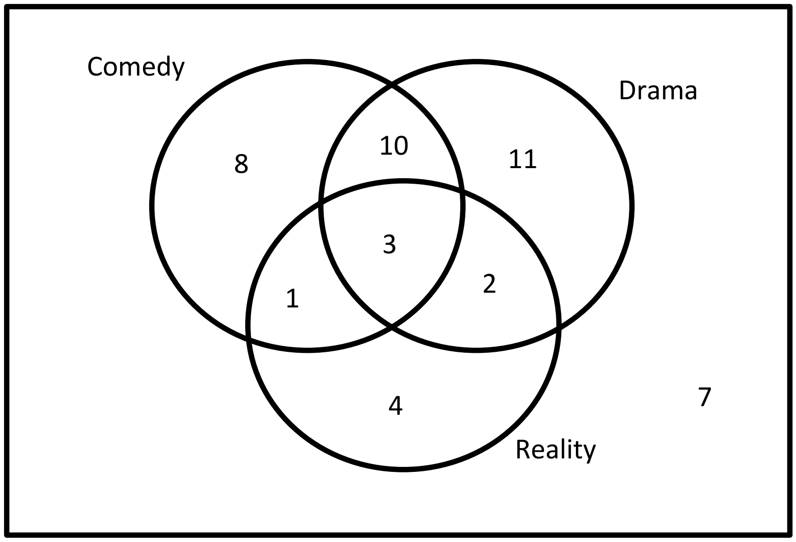
$P(A \cap B) = 0$



Probability Unit – Venn Diagrams

Ex: A group of 100 people were asked what types of television shows they watched. The results are presented in the Venn diagram below.

$\Omega = 46$



Determine the following probabilities:

- A person watches comedy shows

- A person watches drama or reality shows

- A person watches comedy and reality shows

- A person watches comedy and drama shows

- A person watches reality and drama shows

- A person watches comedy, drama, and reality shows

Practice Questions

1) There are 10 friends, Alex, Blaire, Casey, Drew, Erin, Francis, Glen, Hunter, Ira, and Jade.

Alex, Casey, Drew, and Hunter all play soccer.

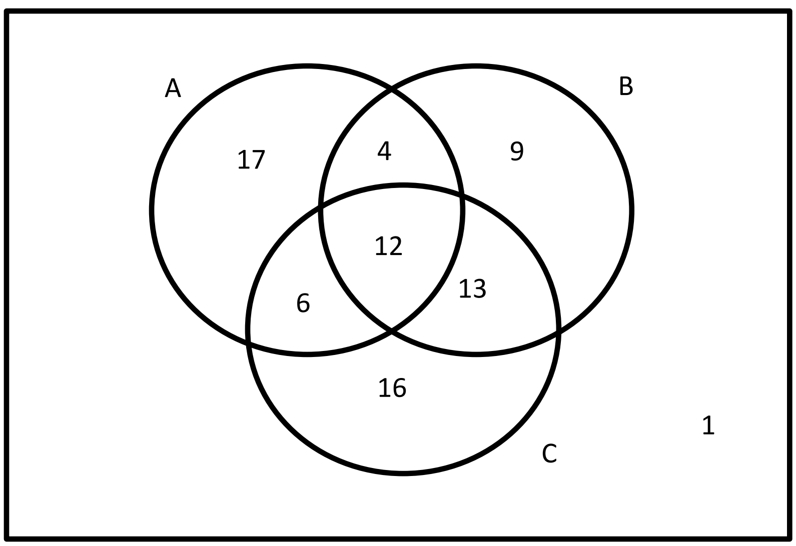
Casey, Drew, and Jade all play tennis.

Drew, Glen, and Jade all play volleyball

Create a Venn diagram to represent this scenario.

2) The Venn diagram below shows the probabilities of event A, event B, and event C.

$\Omega = 78$



Determine the following probabilities:

a) $P(A)$

b) $P(A \cap B)$

c) $P(B \cup C)$

d) $P(A \cap B \cap C)$

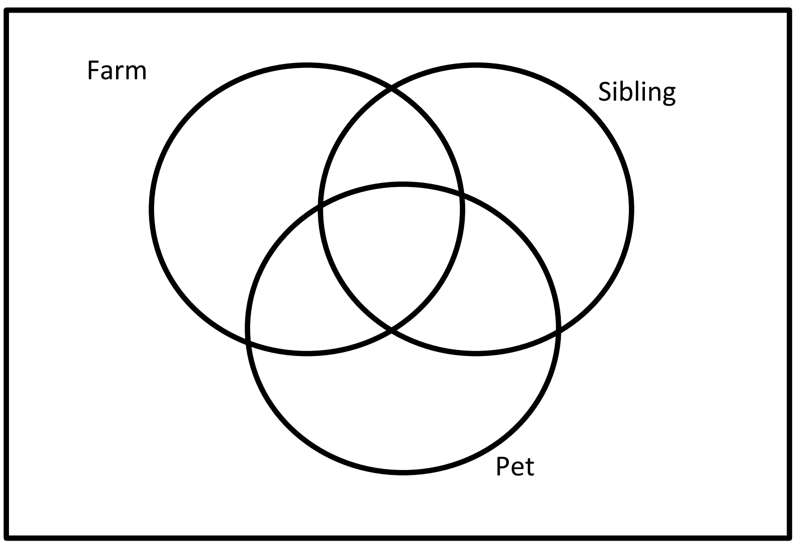
Probability Unit – Completing a Venn Diagram

5.4. CONSTRUCTING VENN DIAGRAMS

Often, before we can use a Venn diagram to determine probabilities, we will need to construct one given a series of clues.

Ex: 35 students were asked 3 questions: 1) Do they live on a farm; 2) Do they have a sibling; 3) Do they have a pet. Using the results below, create a Venn diagram and determine the probability that a student lives on a farm or has a sibling.

- 16 students live on a farm
- 17 students have a sibling
- 28 students have a sibling or have a pet
- 3 students who live on a farm own a pet and have a sibling
- 5 students have a sibling and own a pet
- 11 students live on a farm and have a sibling
- 26 students own a pet or live on a farm
- 1 student does not live on a farm, does not have a sibling, and does not have a pet



Practice Questions

1) 20 students went on an outdoor winter trip. The students could choose to ski, snowshoe, or dogsled. All students participated in at least one activity.

- 10 students went skiing
- 7 students went snowshoeing
- 11 students went dogsledding
- 2 students chose to both ski and snowshoe
- 4 students went snowshoeing and dogsledding
- 3 students went skiing and dogsledding.
- 1 student did all three activities

a) Complete a Venn diagram to represent this scenario.

b) If one student was chosen at random, what is the probability that the student went dogsledding or skiing?

2) An English teacher took a class survey to determine what books the students had read. Of the 24 students in the class:

- 11 students had read The Absolutely True Diary of a Part-Time Indian
- 7 students had read When Everything Feels Like the Movies
- 10 students had read The Hate U Give
- 3 students had read When Everything Feels Like the Movies and The Absolutely True Diary of a Part-Time Indian
- 4 students had read The Absolutely True Diary of a Part-Time Indian and The Hate U Give
- 1 student had read The Hate U Give and When Everything Feels Like the Movies
- 1 student had read all three books

a) Complete a Venn diagram to represent this scenario.

b) If one student was chosen at random, what is the probability that the student did not read any of the books?

5.5 CONTINGENCY TABLES

Contingency tables are useful when pieces of information fall in two categories and each category has subcategories. Let's say question 1 had two options: A and B, and question 2 had 2 options, X and Y. A contingency table allows us to present the results of both questions, as well as how the two questions relate to each other. That is, how many people chose $A \cap X, A \cap Y, B \cap X, B \cap Y$.

Ex: Dr. James asked her class of 25 two questions: 1) Which do you prefer: blue, red, yellow? 2) Do you like summer or winter better? The results are presented in the contingency table below

Season ↓ Colour →	BLUE	YELLOW	RED	TOTAL
SUMMER	7	2	6	15
WINTER	3	6	1	10
TOTAL	10	8	7	25

From a contingency table, we can determine probabilities.

$P(B \cap S) =$ $P(R \cap W) =$ $P(S) =$ $P(Y) =$

Sometimes we will have to complete a contingency table before using it to determine probabilities.

Ex: A school is hosting a trip for grade 10 and grade 11 students. The possible destinations are: Banff, Vancouver, and Halifax. The teachers collected the following information.

	Banff	Vancouver	Halifax	TOTAL
Grade 10				250
Grade 11				
TOTAL	300	225		750

Additionally, the teachers know the following:

- Given that a student is in Grade 10, the probability they choose Banff is $\frac{9}{25}$
- The probability that the student is in Grade 11 given that they chose Vancouver is $\frac{5}{9}$

a) What is the probability that a student, chosen at random, is in grade 10 and choose Halifax?

b) Given that a student chose the Banff trip, what is the probability that they are in Grade 10?

Practice Questions

1) A group of 115 students were surveyed about what board games they would like to play during a game night, and the snacks they would like to eat. The results are presented in the contingency table below.

	Pizza Rolls	Chips and Dip	Cookies	TOTAL
Scrabble	10	3	12	25
Trivial Pursuit	8	14	7	29
Monopoly	14	17	7	38
Risk	12	7	4	23
TOTAL	44	41	30	115

a) If a student was chosen at random, what is the probability they want to play Scrabble and want to eat Cookies?

b) If a student was chosen at random, what is the probability they want to play Risk?

c) If a student was chosen at random, what is the probability they want to eat Pizza Rolls or Chips and Dip

2) A school is planning the schedule for the following year. Students in Grade 11 must choose two options. They must choose an art (Art, Music, or Drama) and another option (Home Economics, Biology, or Law). The students returned their options sheets and we know the following:

- 50 students chose Music
- There are a total of 200 students entering grade 11 next year
- The probability that a student chose Art is $\frac{3}{5}$
- The probability that a student chose Home Economics is 0.3
- Given that a student chose Home Economics, the probability that the student chose Art is $\frac{1}{4}$
- The probability that a student chose Law, given that the student chose Art, is $\frac{2}{3}$
- The probability that a student chose Music, given that the student chose Home Economics is $\frac{5}{12}$
- No students chose both Drama and Law
- The probability a student chose Music and Law is 0.05

a) Complete a contingency table for the above scenario.

b) What is the probability that a student chose Biology?

c) What is the probability that a student chose both Drama and Home Economics?

5.6 CONDITIONAL PROBABILITY

Conditional probability is the probability that an event will occur, given that another event has already occurred. We can determine the conditional probability using a contingency table or a formula.

To use a contingency table, focus attention on the row or column in the "given that" statement. Then only use those numbers to determine the possible ways the event can occur and the total number of possibilities.

Ex: Dr. James asked her class of 25 two questions: 1) Which do you prefer: blue, red, yellow? 2) Do you like summer or winter better? The results are presented in the contingency table below

Season ↓ Colour →	BLUE	YELLOW	RED	TOTAL
SUMMER	7	2	6	15
WINTER	3	6	1	10
TOTAL	10	8	7	25

a) If a student is chosen at random, what is the probability that the student prefers yellow given that the student likes winter best?

b) Given that a student likes Red best, what is the probability they also prefer summer?

If we do not have a contingency table (and do not have enough information to construct one), we can use the following formula to determine conditional probability.

$$P(B\ given\ A) = P(B|A) = \frac{P(A \cap B)}{P(A)}\ \ where\ P(A) \neq 0$$

Or $P(B|A) \times P(A) = P(A \cap B)$

Ex: Dr. James gave her class two tests. 62% of the class passed both tests and 80% of the class passed the first test. What percent of those who passed the first test also passed the second test?

Practice Questions

1) Canadian tourists coming back from a South American trip were interviewed about the countries they visited. Among the 40 tourists in the group:

- 20 visited Argentina
- 30 visited Brazil
- 12 visited both Argentina and Brazil

a) Given that a tourist visited Argentina, what is the probability they also visited Brazil?

b) Given that a tourist visited Brazil, what is the probability they also visited Argentina?

2) A group of 115 students were surveyed about what board games they would like to play during a game night, and the snacks they would like to eat. The results are presented in the contingency table below.

	Pizza Rolls	Chips and Dip	Cookies	TOTAL
Scrabble	10	3	12	25
Trivial Pursuit	8	14	7	29
Monopoly	14	17	7	38
Risk	12	7	4	23
TOTAL	44	41	30	115

Calculate:

a) $P(Scrabble|Cookies)$

b) $P(Pizza\ Rolls|Risk)$

d) $P(Monopoly|Chips\ and\ Dip)$

e) $P(Cookies|Trivial\ Pursuit)$

5.7 RANDOM EXPERIMENT WITH SEVERAL STEPS

Tree diagrams can help us determine the probability of an event when that event has multiple steps.

To create a tree diagram, we map out the possible options for the first step, then the second, etc. until all possible options are included. For example, let's create a tree diagram to map out the options of flipping a coin 3 times:

Step 1: List options (in this case, heads or tails) and associated probabilities

Step 2: Following each of the results in the first step, list all possible options for the second step (in this case, heads or tails)

Step 3: Following each of the results in the second step, list all possible options for the third step (in this case, heads or tails). Continue adding steps and branches to the tree as needed.

The probability of any given outcome is the product of each result needed to obtain that outcome.

$$P(ABC) = P(A) \times P(B) \times P(C)$$

Keep in mind that in this case, the probability of heads is the same as the probability for tails, but that need not need be the case.

FLIP 1	**FLIP 2**	**FLIP 3**	**OUTCOME**
HEADS ($P = 1/2$)	HEADS ($P = 1/2$)	HEADS ($P = 1/2$)	HHH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS ($P = 1/2$)	HHT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
	TAILS ($P = 1/2$)	HEADS ($P = 1/2$)	HTH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS ($P = 1/2$)	HTT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
TAILS ($P = 1/2$)	HEADS ($P = 1/2$)	HEADS ($P = 1/2$)	THH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS ($P = 1/2$)	THT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
	TAILS($P = 1/2$)	HEADS ($P = 1/2$)	TTH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS ($P = 1/2$)	TTT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$

Probability Unit – Random Experiment With Several Steps

Ex: You are drawing marbles from a bag that contains 3 green marbles and 2 blue marbles. After you draw each marble you put it back in the bag. You draw 3 marbles.

a) What is the probability of drawing a green marble on the first draw, followed by a blue marble, followed by a green marble (that is, what is the probability of GBG)?

b) What is the probability of drawing exactly 2 green marbles?

Probability Unit – Random Experiment With Several Steps

Ex: You are drawing marbles from a bag that contains 3 green marbles and 2 blue marbles. After you draw each marble you do not put it back in the bag. You draw 3 marbles.

a) What is the probability of drawing a green marble on the first draw, followed by a blue marble, followed by a green marble (that is, what is the probability of GBG)?

b) What is the probability of drawing 2 green marbles?

Practice Questions

1) A jar contains 5 red marbles, 3 yellow marbles, and 2 black marbles. Three marbles are drawn consecutively. Calculate the following probabilities:

a) one marble of each color is drawn if the marbles are drawn with replacement.

b) two red marbles and one black marble are drawn without replacement.

c) three marbles of the same color are drawn without replacement.

d) two black marbles and one yellow marble are drawn with replacement.

Probability Unit – Permutations and Combinations

5.8 PERMUTATIONS AND COMBINATIONS

Often, when we are talking outside of math class, we use the word combination in two different ways.

- We might say: **"My fruit salad has a combination of apples, grapes, and bananas."** But we could just as easily say: "My fruit salad has a combination of bananas, apples, and grapes" and we would be talking about the same fruit salad. When we list the fruit, **order does not matter**.

- We might say: **"The combination to the safe is 472."** If we said: "The combination to the safe is 247" that would be different. When we give this list of numbers, **order does matter**.

We will distinguish between these two possibilities:

- A **combination** is when **order doesn't matter**.
- A **permutation** is when **order does matter**.

Additionally, for each of these cases, sometimes a repetition will be allowed and sometimes a repetition will not be allowed. For example, maybe the lock is designed so that once a number is used it cannot be used again.

Below we will discuss each of the 4 cases:

	Order?	**Replacement?**	**Formula** ***n*** is the number of things to choose from and we are choosing ***r*** times	**Calculator**
Permutation	It matters	Allowed	n^r	Use exponent button
	It matters	Not Allowed	$\frac{n!}{(n-r)!}$	nPr button
Combination	Doesn't matter	Not Allowed	$\frac{n!}{r!\,(n-r)!}$	nCr button
	Doesn't matter	Allowed	$\frac{(r+n-1)!}{r!\,(n-1)!}$	Use algebra and the ! button on your calculator

Permutations

A **permutation** is an **ordered** combination. There are two types of permutations

- **Replacement is allowed** – In the safe example above, the "combination" to unlock the safe could be 444 – even though the 4 was used as the first number, it can be re-used.

- **Replacement is not allowed** – If there are 3 people running a race and we are interested in the possible ways they might place. If a person came in first, that person could not also come in second or third.

Permutations with Replacement

When repetition is allowed (with replacement) and order matters, the formula is:

$$number\ of\ permutations\ (with\ replacement) = n^r$$

where ***n*** is the number of things to choose from and we are choosing ***r*** times.

Ex: In the lock example above, there are 10 digits to choose from {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and we choose 3 of them.

Calculate the number of possible "combinations" to the lock.

Permutations without Replacement

In this case, there are fewer possibilities, because we cannot re-use the number once it's been used.

In general, when order matters and repetition is not allowed (without replacement), we can determine the number of possibilities using the following:

$$(\#\ of\ elements\ for\ 1st\ position) \times (\#\ of\ elements\ for\ 2nd\ position) \times \ldots \times (number\ of\ elements\ for\ last\ position)$$

We can use something called the factorial function to write this more easily. The factorial function (symbol: !) just means to multiply a series of descending natural numbers. (The ! button is also on most calculators).

Ex: $4! = 4 \times 3 \times 2 \times 1 = 24$

We can write the formula for determining the number of possibilities when order matters and repetition is not allowed (without replacement) using the formula:

$$\frac{n!}{(n-r)!}$$

Where *n* is the number of things to choose from and we are choosing *r* times.

Note: You can also use the nPr button on your calculator.

Ex: A group of 10 people are running a race. How many possible ways could first, second, and third place be awarded?

Combinations without Replacement

For combinations without repetition, think about a lottery where numbers are drawn one at a time and if we have all the numbers that are drawn (regardless of the order), we win!

We can write the formula for determining the number of possibilities when order does not matter and repetition is not allowed (without replacement) using the formula:

$$\frac{n!}{r!\,(n-r)!}$$

Where *n* is the number of things to choose from and we are choosing *r* times.

Note: You can also use the nCr button on your calculator.

Ex: There are 30 students in a classroom and the teacher wants to create one group of 3 students to decorate the bulletin board. How many different possible groups are there?

Combinations with Replacement

For combinations with replacement, think about going for ice cream. The ice cream shop offers 10 different flavors of ice cream and you are going to get 3 scoops in a dish. In this case, maybe you want all one flavor. Maybe you want 2 scoops of 1 flavor and 2 of another. Or maybe you want 3 scoops of different flavors. In this case it won't matter which scoop is dished out first, second, or third, and there is replacement, because the same flavor can be used more than once.

We can write the formula for determining the number of possibilities when order does not matter and repetition is not allowed (without replacement) using the formula:

$$\frac{(r+n-1)!}{r!\,(n-r)!}$$

Where *n* is the number of things to choose from and we are choosing *r* times.

Ex: The ice cream shop offers 10 different flavors of ice cream and you are going to get 3 scoops in a dish. How many different combinations of ice cream could you create? If 3 scoops were randomly selected, what is the probability that all 3 scoops are the same flavor?

Probability Unit – Permutations and Combinations

Practice Questions

1) For an electronics project, Keyla must place 6 LEDs next to one another in a line. Keyla has packages of red, green, yellow, and blue LEDs. How many different possible arrangements could Keyla make?

2) You have placed 15 different songs into your favorites list on your phone. Your phone will randomly generate playlists of 6 songs where no song is repeated within each playlist. How many different playlists can your phone generate?

3) Dr. James is putting together treats for her class. She has purchased bags of five different flavored candies: cherry, grape, orange, green apple, and pink lemonade. Dr. James randomly gives 3 candies to each student.

a) How many different arrangements of candies are possible?

b) What is the probability that a student would receive 3 grape candies?

c) What is the probability that a student would receive 3 candies of the same flavor?

4) 10 marbles are placed in a bag. The marbles are numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. A student is asked to randomly pick 2 marbles simultaneously.

a) How many different arrangements of numbers could the student pick?

b) What is the probability that a student would pick two consecutive numbers (ex: 1 and 2 or 6 and 7)?

Probability Unit – Odds

5.9 ODDS

Odds and probability are related concepts. We must between **odds for** and **odds against**.

The **odds for** a particular event to occur is the ratio of the number of favorable outcomes to the number of unfavorable outcomes.

Odds for Event A
$Odds\ for\ Event\ A = \dfrac{The\ number\ of\ ways\ event\ A\ occurs}{The\ number\ of\ ways\ A\ does\ not\ occur}$ OR $= \dfrac{Number\ of\ favorable\ outcomes}{Number\ of\ unfavorable\ outcomes}$ OR $Number\ of\ favorable\ outcomes: Number\ of\ unfavorable\ outcomes$

Ex: The odds that a football team will win the championship are 3:2. This means that the team has 3 chances of winning and 2 chances of losing.

The **odds against** a particular event to occur is the ratio of the number of unfavorable outcomes to the number of favorable outcomes.

Odds against Event A
$Odds\ against\ Event\ A = \dfrac{The\ number\ of\ ways\ event\ A\ does\ not\ occur}{The\ number\ of\ ways\ A\ occurs}$ OR $= \dfrac{Number\ of\ unfavorable\ outcomes}{Number\ of\ favorable\ outcomes}$ OR $Number\ of\ unfavorable\ outcomes: Number\ of\ favorable\ outcomes$

Ex: The odds against a player who bets on "0" in roulette are 36:1. That means the player has 36 chances of losing and 1 chance of winning.

Probability Unit – Odds

Difference between probability and odds

Probability of Event A: P(A)
$P(A) = \frac{\textit{The number of ways event A can occur}}{\textit{The total number of possible outcomes}}$ $= \frac{\textit{number of favorable outcomes}}{\textit{number of favorable outcomes} + \textit{number of unfavorable outcome}}$

Ex: The odds that the basketball team will win the next game are 4 to 3. What is the probability that the basketball team will NOT win the next game?

Practice Questions

1) The probability that the local basketball team will win the next game is estimated at $\frac{2}{7}$.

a) What are the odds for the team winning the next game?

b) What are the odds against the team winning the next game?

2) Two dice (numbered 1 to 6) are rolled. What are:

a) the odds for obtaining a sum of 7?

b) the odds against obtaining a sum of 12?

c) the odds for obtaining a sum of 2?

d) the odds for obtaining a sum of 1?

5.10 EXPECTATION AND FAIRNESS

We can use mathematical **expectation** to determine how much money a person would expect to win (or lose) when placing a bet, given they know the probability of winning (and of losing). We also use the term "**expected gain**" interchangeably with expectation.

- If the result is positive, the person can expect to make a profit.
- If the result is negative, the person can expect to lose money.

$$Expected\ gain = (probability\ of\ winning) \times \underbrace{(net\ gain)}_{Gain -\ initial\ bet} + (probability\ of\ losing) \times \underbrace{(loss)}_{initial\ bet\ (written\ as\ negative)}$$

Ex: In a game of roulette, a person who has bet on the winning number receives 35 times the amount of the bet. Since the roulette wheel slots are numbered 0 to 36, the probability that the ball will land on a given number is $\frac{1}{37}$.

Determine the expected gain for a bet of $10.

Fairness

- When the mathematical expectation (or expected gain) of a game is greater than 0, it is advantageous to the person playing.
- When the mathematical expectation (or expected gain) of a game is less than 0, it is disadvantageous to the person playing.
- When a game's mathematical expectation is equal to 0, the game is considered **fair**. That is, given a person plays an infinite number of times, they can expect to lose nothing and gain nothing.

Ex: A game consists of drawing a marble at random from a set composed of 5 red marbles and 4 blue marbles. It costs a person \$10 to place a bet. If a blue marble is drawn, the person wins \$12.50 plus their initial bet. If a red marble is drawn, the \$10 bet is lost. Is this game fair?

Ex: For \$2, a person can participate in a draw for a \$50 gift card. The probability of winning the draw is $\frac{1}{100}$. Is this draw fair?

Probability Unit – Expectation and Fairness

Practice Questions

1) A carnival game costs $2 to play. In this game, there is a jar containing 2 black marbles and 3 white marbles. If you draw a black marble, you get $4. If you draw a white marble you get nothing. What is the expected gain from this game?

2) Consider the following game. Toss a coin twice. You lose your initial bet of $10 if you do not get tails on either toss. You win $4 and get your initial bet back if you get tails on one toss. You win $8 and get your initial bet back if you get tails on both tosses. What is the expected gain from this game?

3) In a game at a carnival, a person must draw a card from a deck of 54 cards (the deck contains both jokers). If the person draws one of the four aces, they win a box of candy valued at $10. If it costs $2 to play, is this game fair?

4) In a game at a carnival, a person must draw a card from a deck of 54 cards. If the person draws one of the four aces, they win a box of candy valued at $10. If the person draws one two jokers, they win a teddy bear valued at $40. If the person draws any other card, they do not win anything.

a) If it costs $2 to play, is this game fair?

b) If the game is not fair, how much should the game cost to play in order for it to be fair?

Probability Unit – Weighted Mean

5.11 WEIGHTED MEAN

Remember that the mean of a set of numbers is the average. To determine mean:

$$Mean = \frac{sum\ of\ data\ values}{total\ number\ of\ values}$$

Add all the numbers together and divide by how many numbers there are.

Ex: A student received the following grades on their math quizzes: 67%, 72%, 85%, 49% and 65%. What is the student's average quiz grade in math?

For a weighted mean, some values are more important than others.

To calculate weighted mean:

- Convert the weight of each item from a percent to a decimal (divide by 100)
- Multiply the value of each item by the weight
- Add (values x weight) together

Ex: A student has the following marks in math.

Item	Grade (%)	Weight (%)
Test 1	72%	50
Quiz 1	64%	25
Homework 1	81%	25

What is the student's overall math mark?

Practice Questions

1) In order to calculate final grades in a class, the teacher combines assignment grades, test grades, and quiz grades. Assignments are worth 30% of the final grade, tests are worth 50%, and quizzes are worth 20%. If a student has an assignment grade of 82%, a test grade of 65%, and a quiz grade of 74%, what is the student's final grade?

2) Nico wants to buy a new car and has decided to rate each car according to the following criteria and values:

- Appearance: 15%
- Reliability: 40%
- Efficiency: 15%
- Comfort: 30%

The following is a list of possible cars and the ratings of each (0 is the lowest mark and 10 is the highest).

	Appearance	Reliability	Efficiency	Comfort
Car A	5	9	8	6
Car B	6	10	8	4
Car C	4	7	3	9

Nico has decided to buy the car with the highest average rating. Which car should Nico buy?

5.12 PROBABILITY EXAM STYLE QUESTIONS

Multiple Choice

1) Before going into their last game of the season, the high school rugby team had won five games and lost three games.

Given their record, what are the odds in favor of winning their last game?

A) 5 to 3 C) 5 to 8

B) 3 to 5 D) 3 to 8

2) A gumball machine contains 120 gumballs that are either blue or red.

If a gumball is drawn at random from the machine, the odds that the gumball will be blue are 3 to 5.

How many blue gumballs are there in the gumball machine?

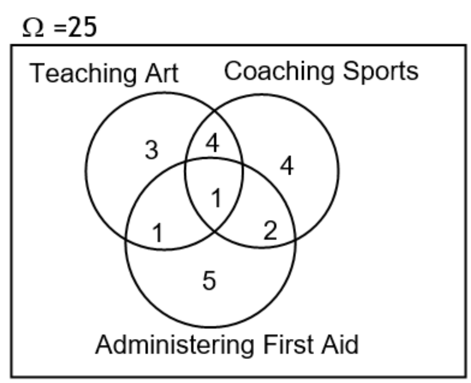
A) 45 C) 72

B) 48 D) 75

3) Counsellors are hired for a large summer camp; 25 students applied for the positions.

To be hired, they must be qualified in one, two or three areas of expertise:

- Teaching art
- Coaching sports
- Administering first aid



The diagram shows the number of students with each expertise.

Which of the following statements is true?

A) The number of students that have expertise in teaching art and coaching sports is 1.

B) The number of students that have expertise in coaching sports or administering first aid is 17.

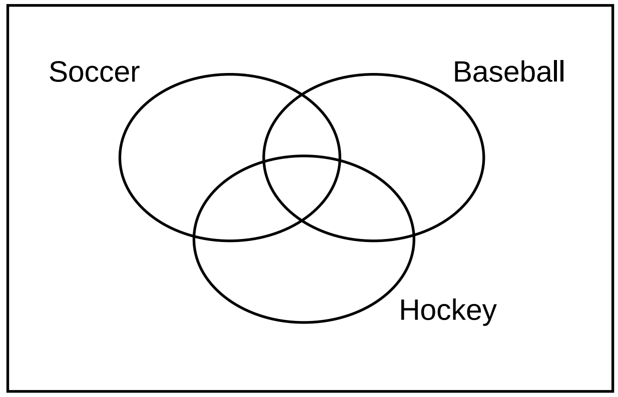
C) The number of students that have expertise in teaching art, administering first aid and coaching sports is 8.

D) The number of students that have no expertise in these areas is 4.

Short Answer

4) A school offers three sports for the Sec V students to play: soccer, baseball and hockey. There are 100 Sec V students. A student can choose to play more than one sport. Here is a list of how many of those students signed up for the sports options:

- Soccer 42
- Baseball 46
- Hockey 45
- All 3 sports 20
- Soccer and baseball 30
- Soccer and hockey 24
- Baseball and hockey 29



a) **How many students signed up only for hockey?**

b) **How many students did not sign up for any sports?**

5) A survey was conducted at a nearby university to learn about the commuting habits of both students and employees. The respondents were asked whether they use public transit or drive a car. Given the respondent is a student, there is a probability of $\frac{4}{7}$ that he or she uses public transport.

The incomplete table below illustrates additional findings of the survey.

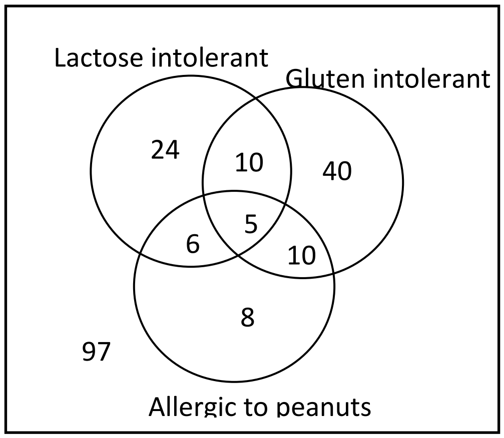
	PUBLIC TRANSIT	CAR	TOTAL
STUDENT			3934
EMPLOYEE			1686
TOTAL	2856	2764	5620

a) **How many students use public transit?**

b) **What is the probability that the respondent is a student, given that the respondent drives a car?**

6) The school nurse asked 200 students if they had any dietary restrictions.

Their responses were recorded in the Venn diagram below.



One of the 200 students was chosen at random.

Given that the chosen student is gluten intolerant, what is the probability that this student is also lactose intolerant?

Long Answer

7) While at a fundraising event, Brian wants to pick a game that will allow him to win a prize. He has done some research and has decided to play either Die Roll or Roulette Wheel.

The amount of money Brian must bet to play either game is the same.

Die Roll

The player rolls an 8 sided die after placing his or her bet.

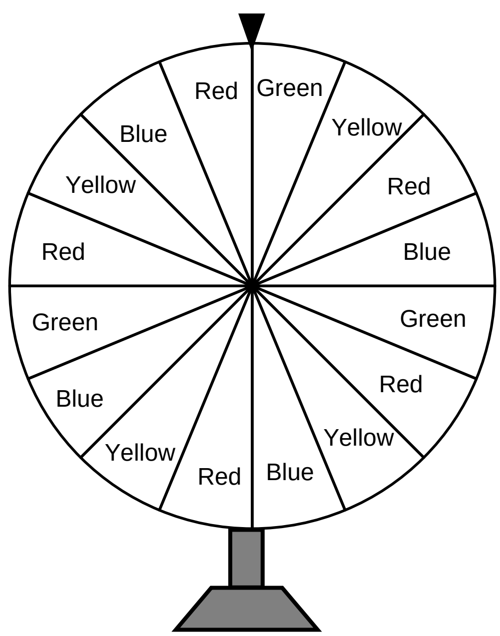
- If an odd number is rolled, the player wins nothing.
- If a 2, 4 or 6 is rolled, the player wins the value of the bet.
- If an 8 is rolled, the player wins $30.

This game is fair.

Roulette Wheel

The player places a bet and then spins the wheel. There are 16 congruent sections on the wheel.

- If the wheel stops on one of the 4 yellow sections, the player wins $6.
- If the wheel stops on one of the 5 red sections, the player wins nothing.
- If the wheel stops on one of the 4 blue sections, the player wins $8.
- If the wheel stops on one of the 3 green sections, the player wins $10.



Which game is more advantageous for Brian to play?

8) A school is offering 3 different trip destinations: Paris, Los Angeles and New York City. Students may only choose only 1 destination. The teachers have collected the following data based on destination and grade.

- There are 750 students who signed up.
- Twice as many grade 10 students as grade 11 students signed up.
- The probability that a student signed up for Paris is $\frac{2}{5}$.
- The probability that a student signed up for Los Angeles is the same as the probability that they signed up for New York City.
- Given that a student chose Paris, the probability that the student is in Grade 11 is 30%.
- Given that a student is in Grade 10, the probability that they signed up for Los Angeles is 0.25.

Given that a student chosen at random is in Grade 11, what is the probability that they signed up for New York City?

9) While at a carnival, Serena finds two games of chance that will allow her to win a prize. She decided to play either The Colour Is Right or The Fruit Wheel.

The amount of money Serena must bet to play either game is the same.

THE COLOUR IS RIGHT

After placing a bet, the player randomly selects a marble from a bag.

The bag contains 5 red marbles, 2 blue marbles and 3 yellow marbles.

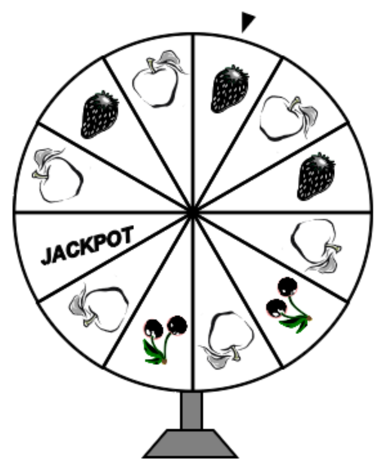
- If a red marble is chosen, the player does not receive any money.
- If a blue marble is chosen, the player receives $28.
- If a yellow marble is chosen, the player receives the value of the bet.

This game is fair.

THE FRUIT WHEEL

This game consists of spinning a wheel after placing a bet. The wheel is divided into 12 congruent sectors.

- If the pointer stops on an apple, the player receives $4.
- If the pointer stops on a strawberry, the player does not receive any money.
- If the pointer stops on a cherry, the player receives $12.
- If the pointer stops the JACKPOT section, the player receives $40.



Is The Fruit Wheel game to Serena's advantage, to Serena's disadvantage, or is it fair? Justify your reasoning.

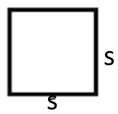
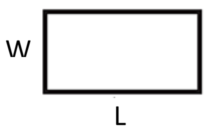
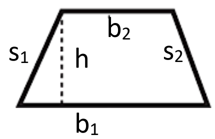
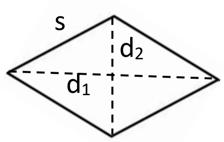
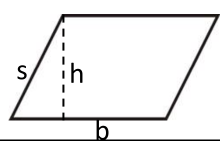
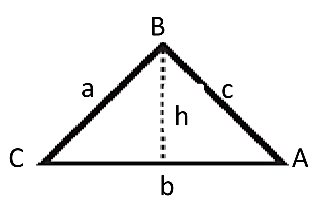
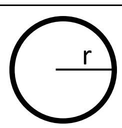
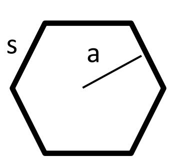
Geometry Unit – Area and Perimeter

6.1 AREA AND PERIMETER

When looking at 2-dimensional shapes, we can talk about perimeter and area.

Perimeter is the distance around the outside of the shape, or the distance to trace the outline of the shape.

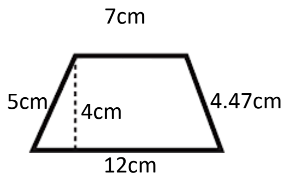
Area is the measure of the inside of the shape – if you were going to color in the shape, the amount of space you would have to color.

Shape	Area	Perimeter
Square 	$A = s^2$	$P = 4s$
Rectangle 	$A = L \times W$	$P = 2W + 2L$
Trapezoid 	$A = \frac{(b_1 + b_2) \times h}{2}$	$P = b_1 + b_2 + s_1 + s_2$
Rhombus 	$A = \frac{d_1 \times d_2}{2}$	$P = 4s$
Parallelogram 	$A = b \times h$	$P = 2b + 2s$
Triangle 	$A = \frac{b \times h}{2}$ $A = \frac{a \times b \times \sin C}{2}$ $A = \sqrt{s(s-a)(s-b)(s-c)}$ Where s is the half the perimeter	$P = a + b + c$
Circle  $d = 2r$	$A = \pi r^2$	$P = 2\pi r$ or $P = \pi d$
Regular "-agon" (all sides are same length) (ex: pentagon, hexagon, heptagon, octagon, nonagon, decagon, etc.)  Where a is the apothem	$A = \frac{P \times a}{2}$	$P = s \times (\# of sides)$

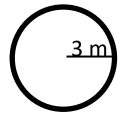
Geometry Unit – Area and Perimeter

Ex: Find the area the perimeter of the following shapes.

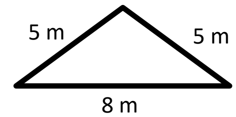
a)



b)



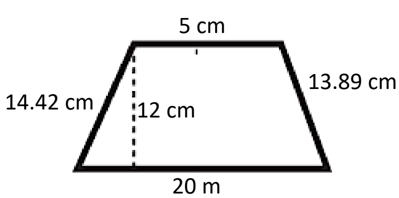
c)



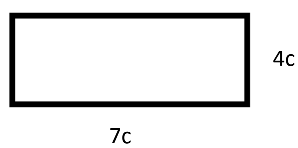
Practice questions

1. Find the area and perimeter of the following shapes

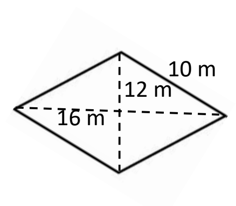
a)



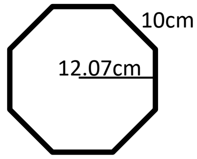
b)



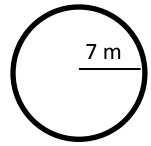
c)



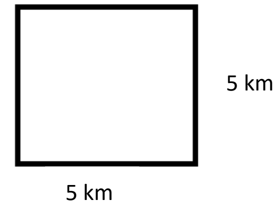
d)



e)



f)



Geometry Unit: Triangles

When working with triangles, we might not be given all the information necessary to determine the area or the perimeter, but we have some tools to help us determine the information we need.

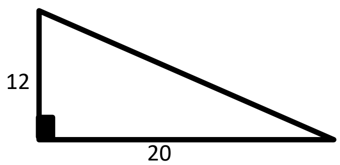
Using the Pythagorean Theorem

If we know 2 sides of a right triangle, we can find the third side by using the Pythagorean Theorem.

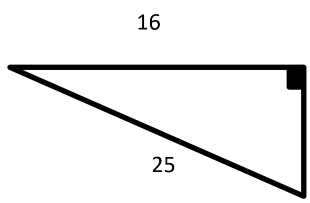
$$a^2 + b^2 = c^2$$

Where c^2 is always the hypotenuse (the longest side, which is across from the right angle)

Ex: Find the missing side in the right triangle below.



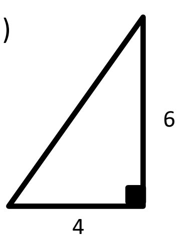
Ex: Find the missing side length in the right triangle below.



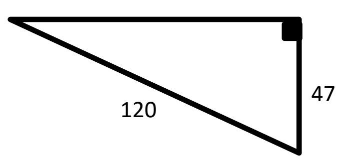
Practice Questions

1. Find the missing side lengths in the triangles below.

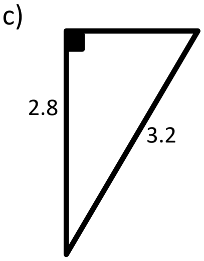
a)



b)



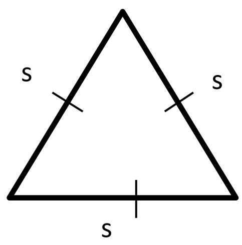
c)



Geometry Unit: Triangles

If we have an equilateral triangle (all the sides are the same length) or an isosceles triangle (2 sides are the same length), we can find the height of the triangle, which is useful in determining the area.

Equilateral Triangle

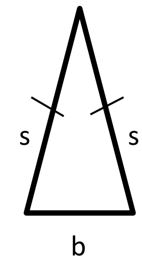


To find the height:

$$h = \sqrt{s^2 - \left(\frac{1}{2}s\right)^2}$$

where s is the length of a side

Isosceles Triangle



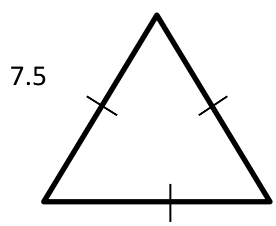
To find the height:

$$h = \sqrt{s^2 - \left(\frac{1}{2}b\right)^2}$$

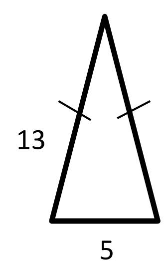
where s is the length of a side
and b is the length of the base

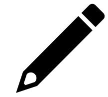
Ex: Find the height of the triangles below.

a)



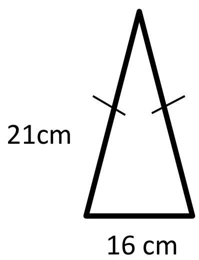
b)



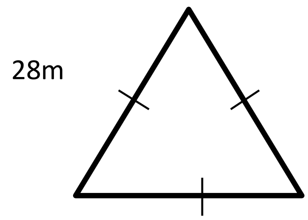


2) Find the heights of the following triangles.

a)

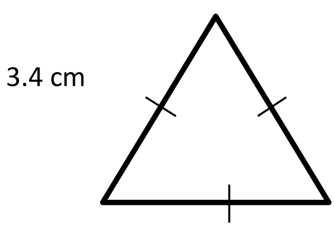


b)

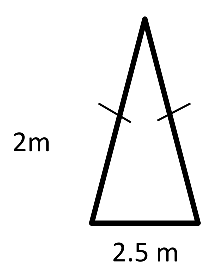


3) Find the area of the following triangles.

a)



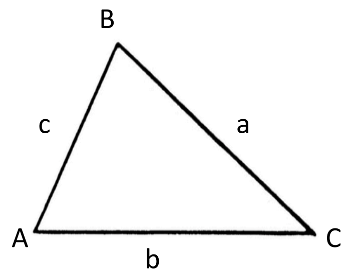
b)



Geometry Unit: Triangles

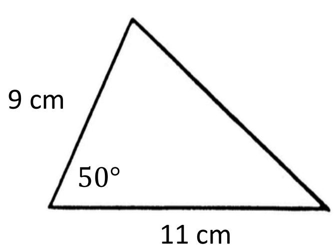
Cosine Law

When we need to find the missing side or missing angle of a triangle that is not a right-angled triangle, we can use **cosine law**.

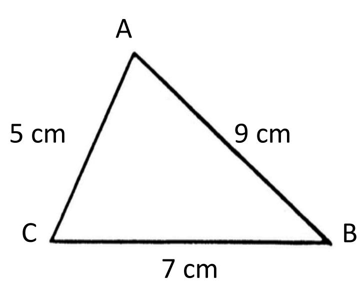


To find a side length	To find an angle
$a^2 = b^2 + c^2 - 2bc\cos A$ $b^2 = a^2 + c^2 - 2ac\cos B$ $c^2 = a^2 + b^2 - 2ab\cos C$	$A = cos^{-1}\left(\frac{a^2 - b^2 - c^2}{-2bc}\right)$ $B = cos^{-1}\left(\frac{b^2 - a^2 - c^2}{-2ac}\right)$ $C = cos^{-1}\left(\frac{c^2 - a^2 - b^2}{-2ab}\right)$

Ex: Find the missing side of the triangle below.

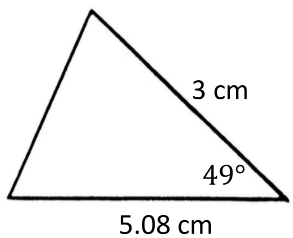


Ex: Find the missing angles of the triangle below.

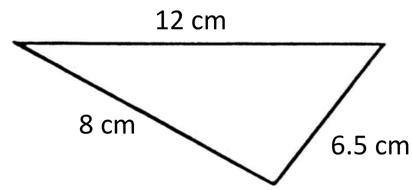


4) Calculate the unknown measurements in each triangle below.

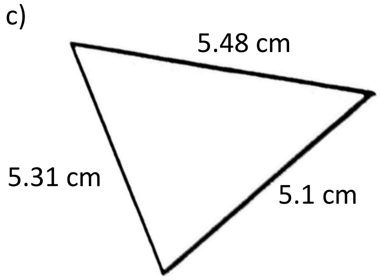
a)



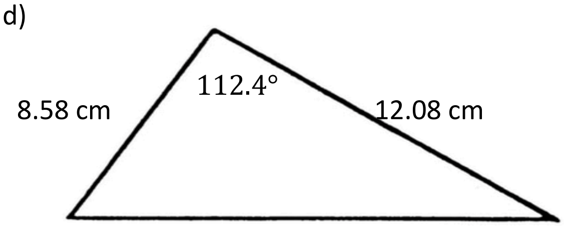
b)



c)



d)



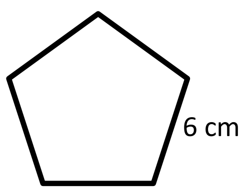
6.3 FINDING THE APOTHEM

If we have a regular "-agon" and we are not given the measurement of the apothem, we can find it if we know the side length.

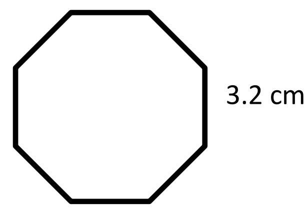
$$Apothem = \frac{side\ length \div 2}{\tan((360 \div \#\ of\ sides) \div 2)}$$

Ex: Find the length of the apothem in each of the regular polygons below.

a)



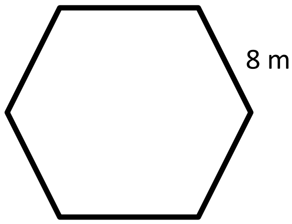
b)



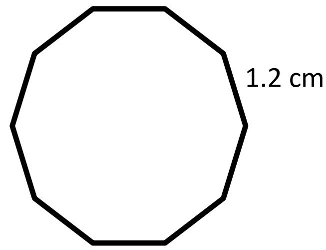
Practice Questions

1) Find the apothem of the following regular polygons.

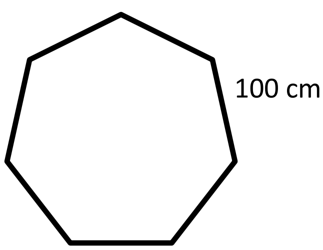
a)



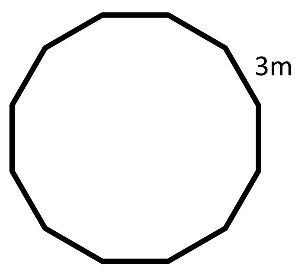
b)



c)



d)



6.4 MISSING MEASURES

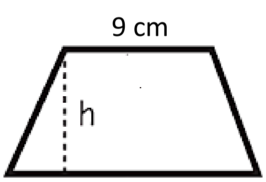
Instead of using the formulas we've just been working with to find area, perimeter, height, or apothem, we can use the same formulas to find missing measures.

To do this, we need to do the following:

1) Choose a formula based on what we know and what we want to find.
2) Plug in what we know.
3) Use algebra to solve for the missing variable.

Ex: A triangle has an area of 12 cm^2 and a base of 2cm. What is the height of the triangle?

Ex: What is the height of the trapezoid below?



$A = 92\ cm^2$

Practice Questions

1) Find the missing measures of the shapes below.

a) A circular table has a circumference of 24.93 ft

What is the radius?

What is the diameter?

What is the area?

b) A circular dartboard has an area of 333.12 in^2.

What is the radius?

What is the diameter?

What is the circumference?

c) An isosceles triangle has a height of 12 m and an area of 48 m^2.

What is the base of the triangle?

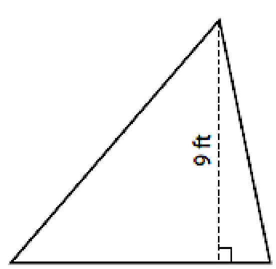
What is the perimeter of the triangle?

d) An isosceles triangle has a base of 16 m and an area of 176 m^2.

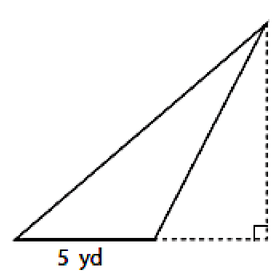
What is the height of the triangle?

What is the perimeter of the triangle?

e) What is the base of the triangle shown below if the area of 36 ft^2?



f) What is the height of the triangle shown below if the area is 25 yd^2?

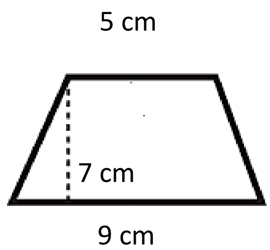


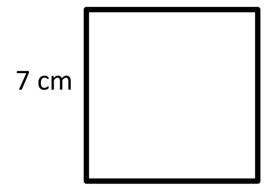
Geometry Unit: Equivalent Figures

6.5 EQUIVALENT FIGURES

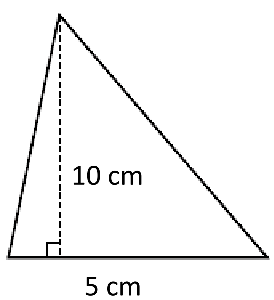
Two figures are considered **equivalent** if they have the same area.

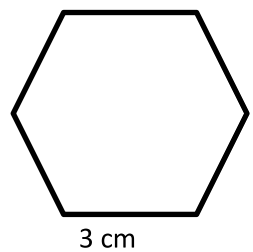
Ex: Show that the following figures are equivalent.



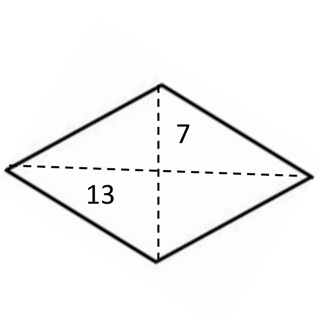


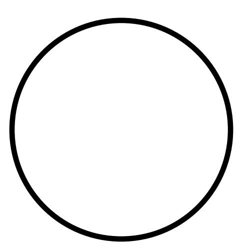
Ex: Are the following figures equivalent?





Ex: Given that the following figures are equivalent, what is the radius of the circle?





Practice Questions

1) A triangle has a base of 9 cm and a height of 8 cm. What is the side length of a square that is equivalent to the triangle?

2) In each of the following cases, find the measure x of the side of the square equivalent to:

a. An 8 cm by 12 cm rectangle:

b. A right triangle with sides measuring 3 cm, 6 cm and 6.7 cm:

c. A trapezoid with a big base of 12 cm, a small base of 4 cm and a height of 8 cm:

d. A rhombus with diagonals measuring 5 cm and 10 cm.

6.6 RULES OF EQUIVALENCY

Remember: a polygon is considered "**regular**" if all the sides have the same length.

Rules of Equivalency:

1) If you have equivalent polygons with the same number of sides, the regular polygon will always have the smallest perimeter.

2) If you have polygons with the same number of sides and the same perimeter, the regular polygon will always have the greatest area.

3) If you have equivalent regular polygons, the one with the greatest number of sides will always have the smallest perimeter.

4) If you have regular polygons with the same perimeter, the one with the most number of sides will always have the greatest area.

Ex: A regular triangle, square, and pentagon all have the same perimeters. Which one has the largest area?

Ex: Using 100 m of fence, a farmer wants to make a rectangular enclosure for a herd of sheep. What is the maximum area of this enclosure?

Practice Questions

1) An equilateral, isosceles and a scalene triangle all have the same perimeter. Which one has the greatest area?

2) A regular pentagon and hexagon are equivalent. Which one has the smallest perimeter?

3) A regular pentagon and hexagon have the same perimeter. Which one has the greatest area?

4) What is the maximum area of a rectangle with a perimeter of 16 cm?

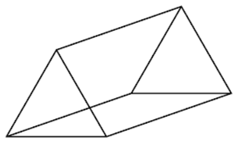
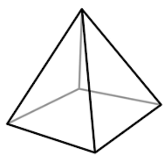
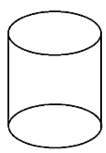
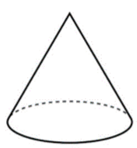
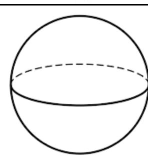
5) What is the maximum area, to the nearest hundredth, of a triangle with perimeter of 12cm?

6.7 LATERAL AREA, SURFACE AREA, AND VOLUME

When considering 3-dimentional solids, we can talk about area (surface and lateral) and volume.

We can think of area as the amount of wrapping paper it would take to cover the object. For surface area we want to cover all the surfaces. For lateral area we leave the top and bottom uncovered.

We can think of volume as the amount of space inside the object.

Solid	Lateral Area	Surface Area	Volume
Prisms 2 bases that can be any shape except circles 	$A_l = P_b \times h$	$A_s = 2A_b + A_l$	$V = A_b \times h$
Pyramids 1 base that can be any shape except a circle 	$A_l = \dfrac{P_b \times sl}{2}$	$A_s = A_b + A_l$	$V = \dfrac{A_b \times h}{3}$
Cylinders Like a prism with a circular base 	$A_l = 2\pi \times r \times h$	$A_s = 2A_b + A_l$	$V = A_b \times h$
Cones Like a pyramid with a circular base 	$A_l = 2\pi \times r \times sl$	$A_s = A_b + A_l$	$V = \dfrac{A_b \times h}{3}$
Spheres 	None	$A_s = 4\pi r^2$	$V = \dfrac{4}{3}\pi r^3$

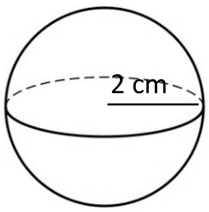
A_l: *Lateral area* A_s: *Surface area* A_b: *Area of base* P_b: *Perimeter of base*

sl: *Slant height* h: *Height* r: *Radius*

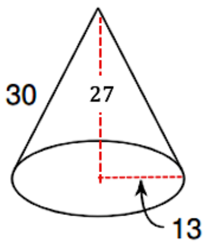
Geometry Unit – Lateral Area, Surface Area, and Volume

Ex: Determine the lateral area, surface area, and volume of each of the following figures.

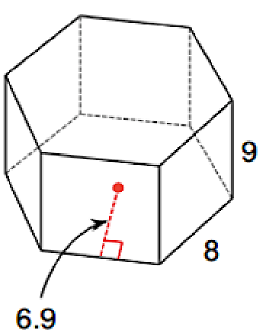
a)



b)

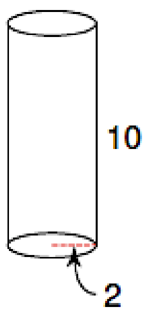


c)

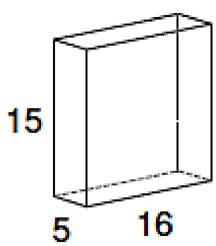


1) Determine the lateral area, surface area, and volume of each of the following figures.

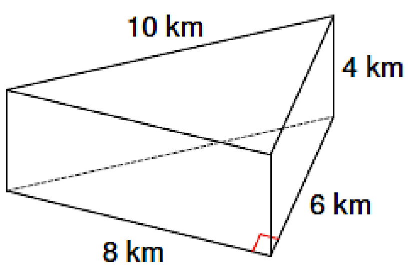
a)



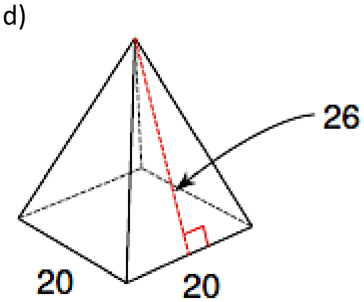
b)



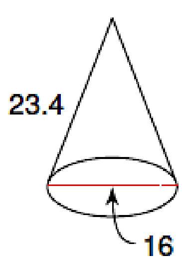
c)



d)



e)



Geometry Unit: Missing Measures

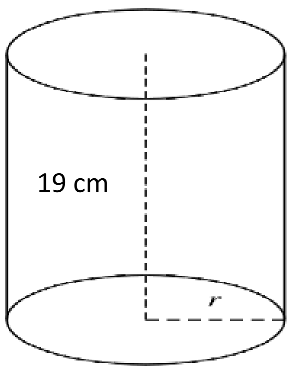
6.8 MISSING MEASURES

Just like with 2-dimensional figures, we can use the formulas we've been working with to find lateral area, surface area, and volume to find missing measures.

To do this, we need to do the following:

1) Choose a formula based on what we know and what we want to find.
2) Plug in what we know.
3) Use algebra to solve for the missing variable.

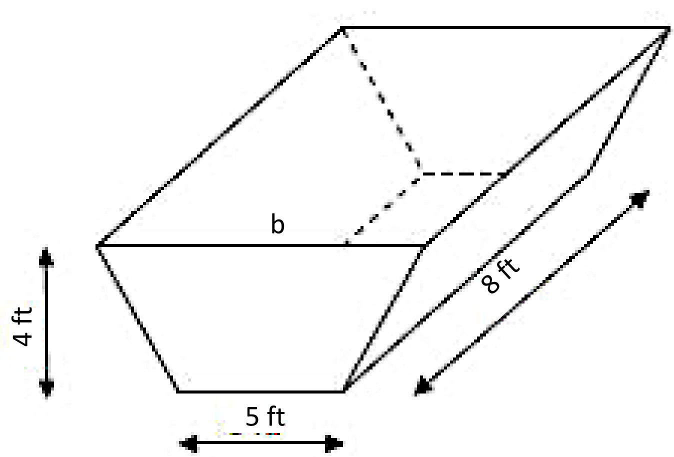
Ex: The cylinder below has a volume of 238.761 cm^3. What is the radius?



Ex: A square-based pyramid has a surface area of 172.8 cm^2. The square base has sides measuring 6 cm. What is the slant height?

Practice Questions

1) Find the missing base, b, given that the volume is 176 ft^2.



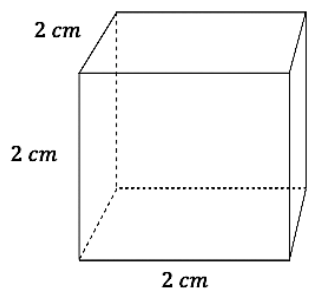
2) A cylinder has a volume of approximately 314 cm^3 and a radius of 5 cm. Solve for the height of the cylinder.

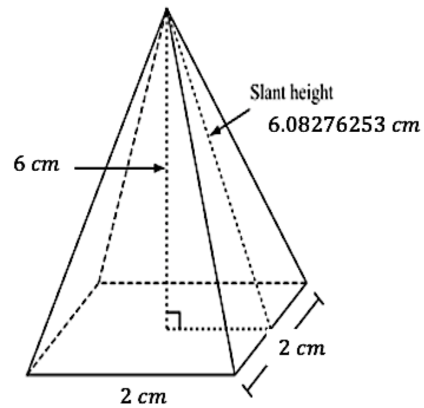
3) A cone has a volume of 37.68 mm^3 and a height of 9 mm. Find the radius.

6.9 EQUIVALENT SOLIDS

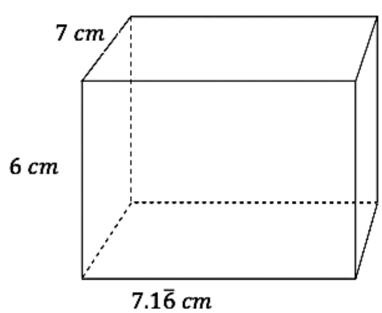
Solids are **equivalent** if they have the same volume. The shape, lateral area, and surface area does not matter.

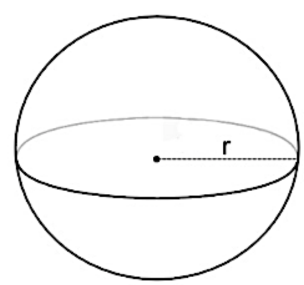
Ex: Are the two following solids equivalent?





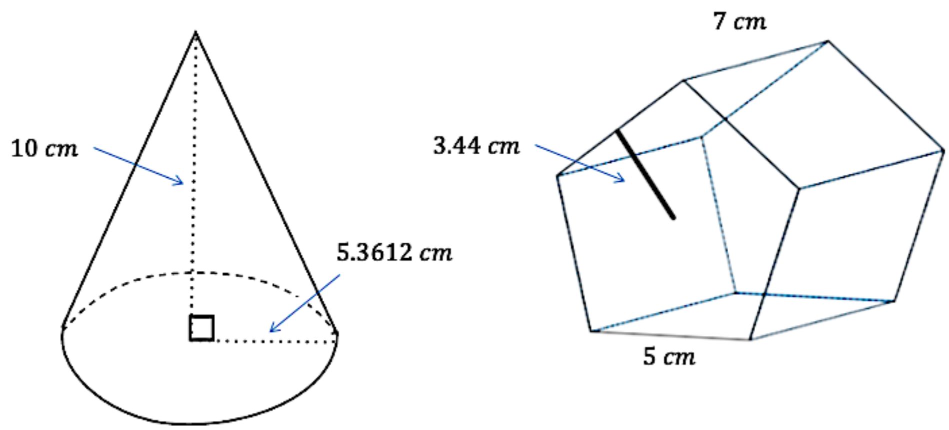
Ex: Given that the two following solids are equivalent, what is the radius of the sphere?





Practice Questions

1) Are the two solids equivalent?



2) A prism with a height of 4 cm has a rectangular base with dimensions 6cm by 9cm. What is the measure of a cube's edge that is equivalent to the prism?

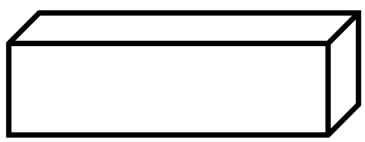
3) A cone and a cylinder are equivalent. The radius and the height of the cone measure 6 cm and 10 cm respectively. What is the height of the cylinder if its radius measures 5 cm?

6.10 RULES OF EQUIVALENCY

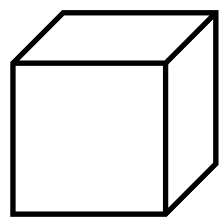
Just like we have rules of equivalency for two-dimensional shapes, there are rules of equivalency for three-dimensional solids.

1) If you have equivalent solids with the same number of faces, the regular solid will always have the smallest surface area.
2) If you have solids with the same number of faces and the same surface area, the regular solid will have the greatest volume.
3) If you have regular equivalent solids, the one with the greatest number of faces will always have the smallest surface area.
4) If you have regular solids with the same surface area, the one with the most faces will have the greatest volume.

Ex: Given the following solids have the same number of faces and the same surface area, which one will have the greatest volume?



Rectangular Prism



Cube

Ex: Of all solids with a total volume of 8 cm^3, what is the shape of the solid with the smallest total area?

Practice Questions

1) What is the maximum volume of a rectangular prism with a total surface area of 150 cm^2?

2) What is the minimum surface area of a rectangular prism with a total volume of 512 cm^3?

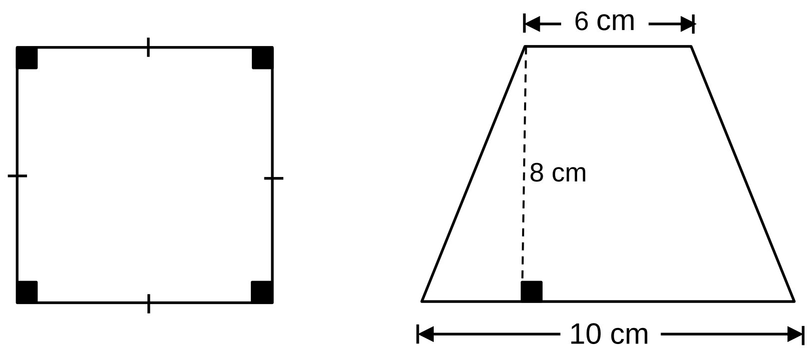
3) What is the minimum area of a solid with a total volume of 113 cm^3

Geometry Unit – Exam Style Questions

6.11 Geometry Exam Style Questions

Multiple Choice

1) The square and the isosceles trapezoid below are equivalent.



What is the perimeter of the square?

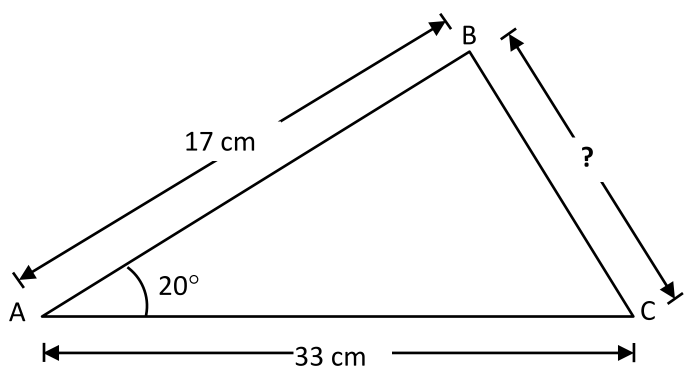
A) 8 cm

B) 32 cm

C) 45.25 cm

D) 64 cm

2) Given the triangle ABC shown below.



What is the measure of segment BC to the nearest unit?

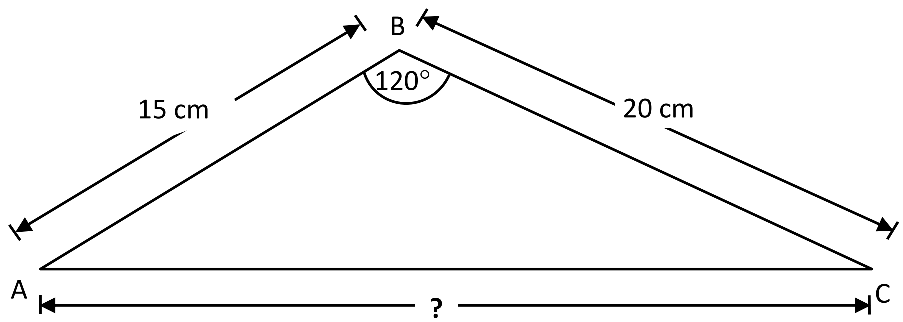
A) 11 cm

B) 15 cm

C) 18 cm

D) 28 cm

3) Consider the triangle ABC below.



To the nearest unit, what is the measure of segment AC?

A) 18 cm

B) 25 cm

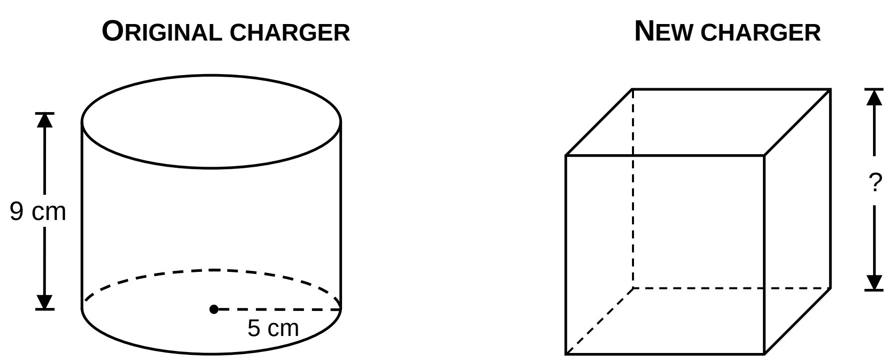
C) 28 cm

D) 30 cm

Short Answer

4) A computer manufacturer is changing the shape of its laptop computer battery chargers. The original charger was in the shape of a right cylinder and had a height of 9 cm and a radius of 5 cm.

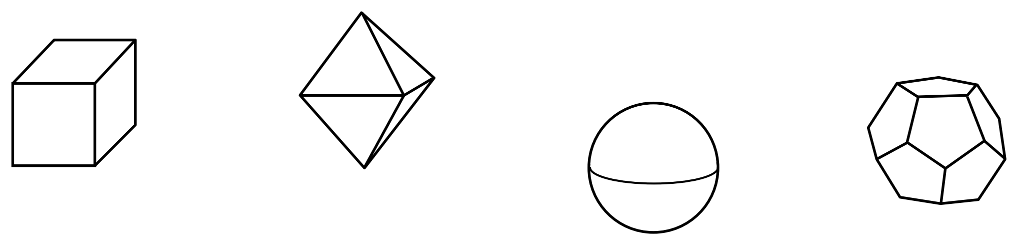
The new charger will be equivalent to the original charger and will be in the shape of a cube.



To the nearest tenth, what is the measure of the new charger's side length?

5) Tim owns a small candy shop called *Candies Made For You*. Today he will make 4 different equivalent solids of candy coated chocolate. Tim knows that the amount of candy coating for each solid will be different. The following are the solids he will be making:

Cube	Regular Octahedron	Sphere	Regular Dodecahedron



List the 4 solids, in <u>increasing</u> order, according to the amount of candy coating needed.

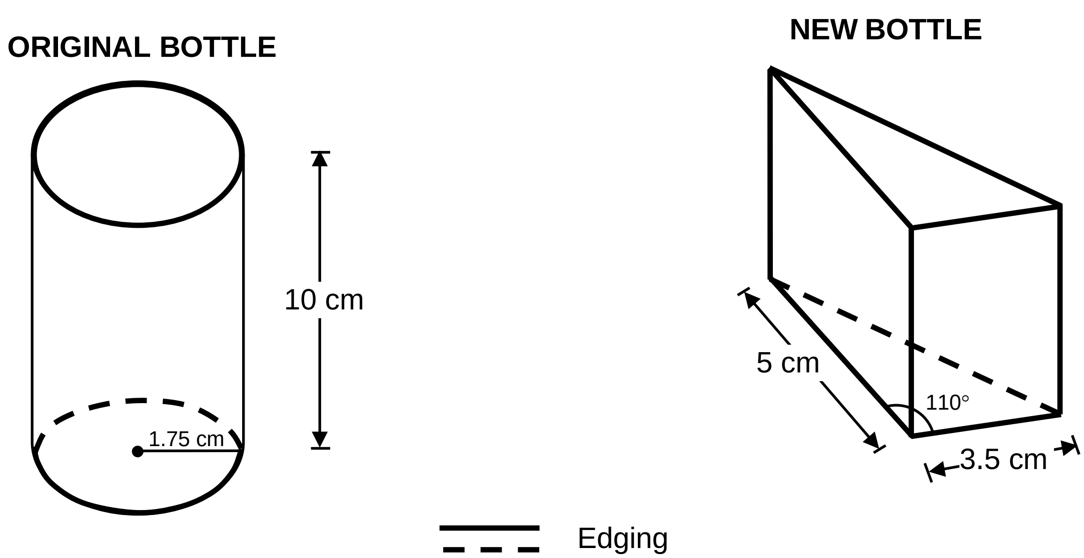
Long Answer

6) A marketing firm has been hired to redesign a bottle used for an exclusive shampoo.

The original design is a right cylindrical shaped bottle with a strip of gold edging along its upper and lower edges. The bottle has a radius of 1.75 cm and a height of 10 cm.

The new bottle is equivalent to the original one. It's in the shape of a right triangular based prism with sides measuring 5 cm and 3.5 cm and a contained angle of 110º. The bottle has gold edging along every one of its edges, without overlapping.

The original and new bottles and their given dimensions are illustrated below:



The marketing firm was required to meet the following criteria in designing this new bottle:

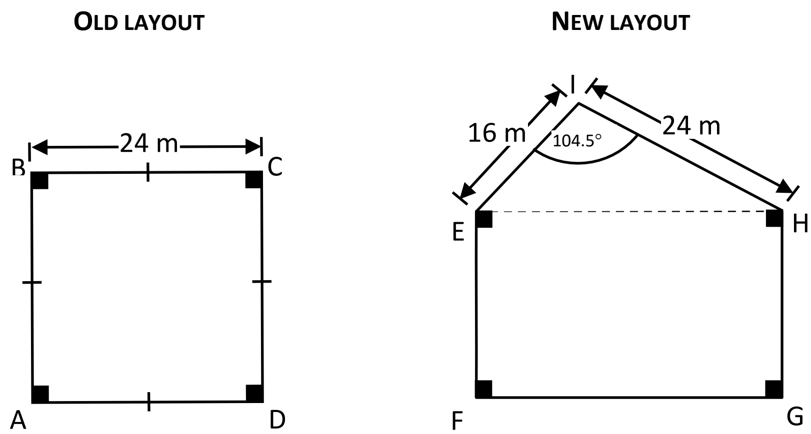
- The length of gold edging on the new bottle must be at most 2.5 times the length of the edging of the original bottle.

Did the marketing firm meet the criteria?

Justify your decision.

7) The administration at Little Treasures, a local daycare, rearranged the layout of its play area

As shown below, the shape of the play area was modified from a square (ABCD) to an equivalent rectangle (EFGH). A triangular-shaped area (EIH) was also added to the new layout.



The daycare will install the following:

- artificial turf on the entire surface of the new play area
- a fence along the perimeter of the new play area

Artificial turf costs \$50 per square metre and is purchased by the square metre. The fence costs \$200 per metre.

To the nearest dollar, what is the total cost of the artificial turf and the fence?

Answer Key: Review Unit

R.1 SOLVING ALGEBRAIC EQUATIONS ANSWERS

1a) $x = 16$	**1b)** $x = -19$	**1c)** $x = 9$	**1d)** $x = 72$
2a) $x = 3$	**2b)** $x = 245$	**2c)** $x = 40$	**2d)** $x = 3$
3a) $x = 2$	**3b)** $x = 5$	**3c)** $x = 5$	**3d)** $x = 2$
4a) $x = 1$	**4b)** $x = -13.5$	**4c)** $x = 0.93$	
5a) $x = 7$	**5b)** $x = -1$	**5c)** $x = -0.71$	

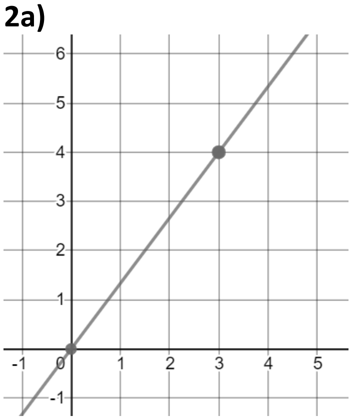
6) $x = 2$	**7)** $x = 8$
8) $x = -0.29$	**9)** $x = 1.64$
10) $x = 8$	**11)** $x = -0.47$
12) $x = 1.25$	**13)** $x = 4$

R.2 LINEAR EQUATIONS ANSWERS

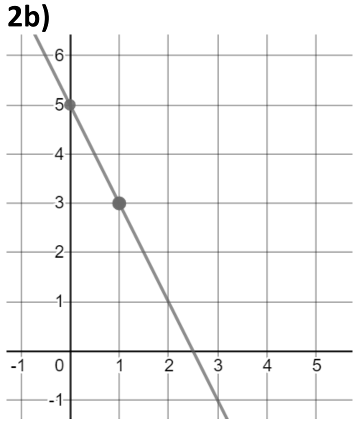
1a) slope: $-\frac{3}{4}$
y-intercept: 3

1b) slope: -5
y-intercept:2

2a)

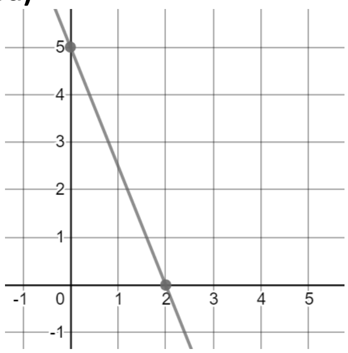


2b)

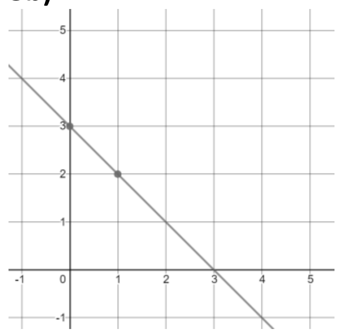


Answer Key: Review Unit

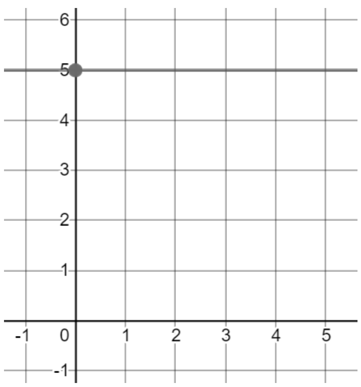
3a)



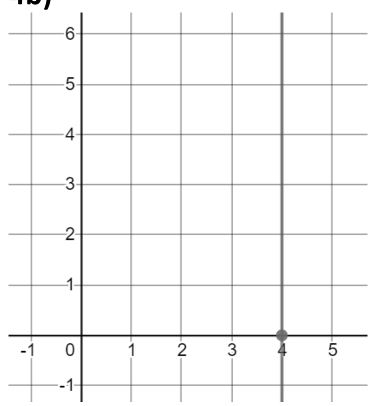
3b)



4a)



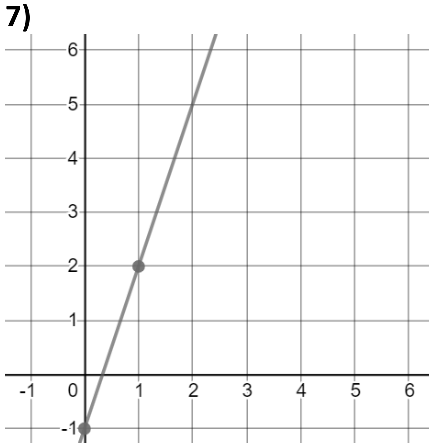
4b)



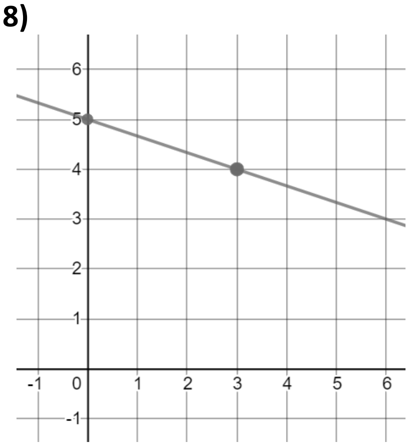
5) slope: $\frac{2}{3}$
y-intercept: -10

6) slope: $\frac{3}{7}$
y-intercept: -5

7)

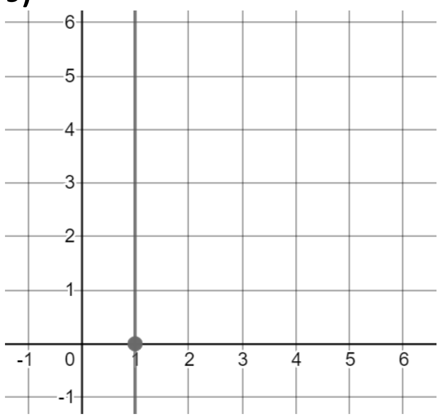


8)

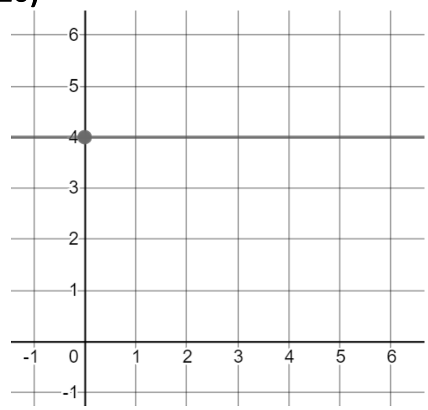


Answer Key: Review Unit

9)

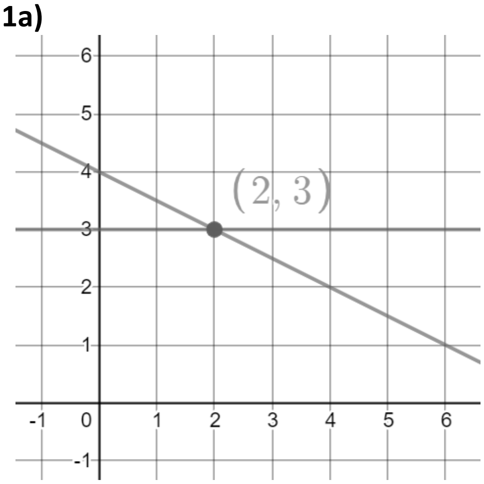


10)



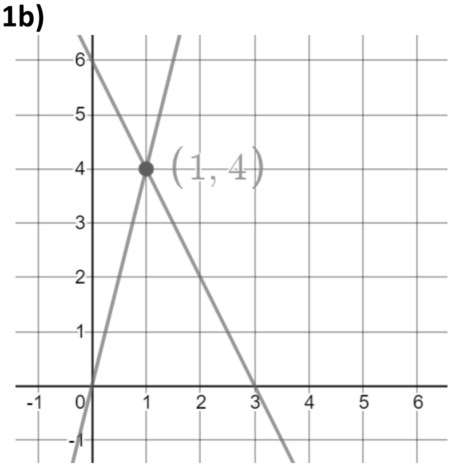
R.3 LINEAR SYSTEMS ANSWERS

1a)



The solution is $(2, 3)$

1b)



The solution is $(1, 4)$

2a) The solution is $(3, -2)$

2b) The solution is $(2, -3)$

3a) The solution is $(1, 7)$

3b) The solution is $(2, 3)$

4a) The solution is $(10, 5)$

4b) The solution is $(4, 4)$

5) The solution is $(2,3)$

6) The solution is $(1,1)$

7) The solution is $(-1, 2)$

8) The solution is $(-4, 2)$

9) The solution is $(6, -7)$

10) The solution is $(1, -2)$

11) The solution is $(0, -5)$

12) The solution is $(3, -8)$

Answer Key: Optimization Unit
1.1 DEFINITIONS AND STEPS

No questions

1.2 OPTIMIZATION STEP 1 ANSWERS

1a)
Let x be the number of pens

Let y be the number of pencils

1b)
Let x be the number of cars

Let y be the number of trucks

1c)
Let x be the number of red tulips

Let y be the number of white tulips

1.3 OPTIMIZATION STEP 2 ANSWERS

1a)
Let x be the number of roses

Let y be the number of tulips

$$x + y = 300$$

1b)
Let x be the number of cookies

Let y be the number of cupcakes

$$x + y = 250$$

1c)
Let x be the number of stuffed dinosaurs

Let y be the number of stuffed bears

$$x + y = 25$$

2a)
Let x be the number of road bikes

Let y be the number of mountain bikes

$$x = 3y$$

2b)
Let x be the number of teas

Let y be the number of coffees

$$x = y + 3$$

2c)
Let x be the number of goats

Let y be the number of chickens

$$y = 5x$$

3a)
Let x be the number of bracelets

Let y be the number of necklaces

$$x \geq y + 5$$

3b)
Let x be the number of citrus soaps

Let y be the number of lavender soaps

$$x + y < 100$$

3c)
Let x be large jars of honey

Let y be small jars of honey

$$y \leq 2x$$

4) Let x be the number of cars and let y be the number of trucks

a) $x + y \leq 200$

b) $x + y \geq 100$

c) $x < 2y$

d) $x \geq 40$

e) $y \leq 150$

Answer Key: Optimization Unit

1.4 OPTIMIZATION STEP 3 ANSWERS

1a)
Let x be the number of cookies

Let y be the number of cupcakes

$$y \leq -x + 75$$

1b)
Let x be the number of dinosaurs

Let y be the number of bears

$$y \leq \frac{1}{3}x$$

1c)
Let x be the number of dinosaurs

Let y be the number of bears

$$y < \frac{1}{2}x$$

2) Let x be the number of teas and let y be the number of coffees

a) $y \leq -x + 700$

b) $y \geq 350$

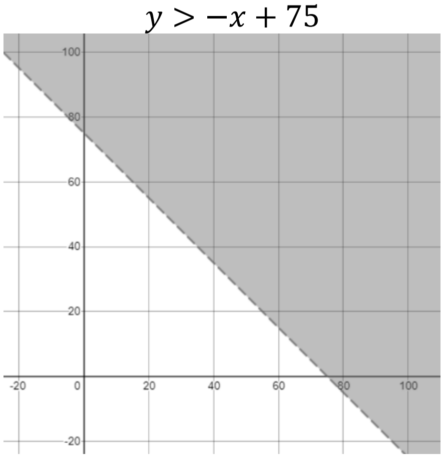
c) y$< 2x$

d) $x \leq 200$

1.5 OPTIMIZATION STEP 4 ANSWERS

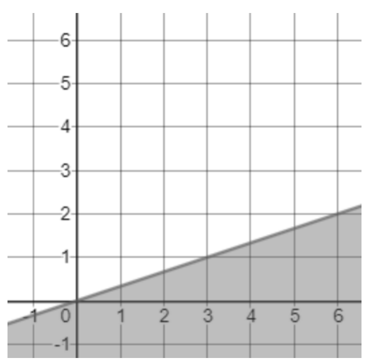
1a)
Let x be the number of cookies
Let y be the number of cupcakes

$$x + y > 75$$
$$y > -x + 75$$



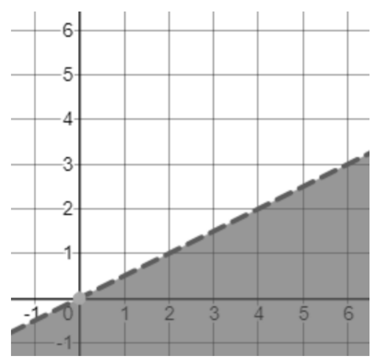
1b)
Let x be the number of dinosaurs
Let y be the number of bears

$$3y \leq x$$
$$y \leq \frac{1}{3}x$$



1c)
Let x be the number of dinosaurs
Let y be the number of bears

$$y < \frac{1}{2}x$$



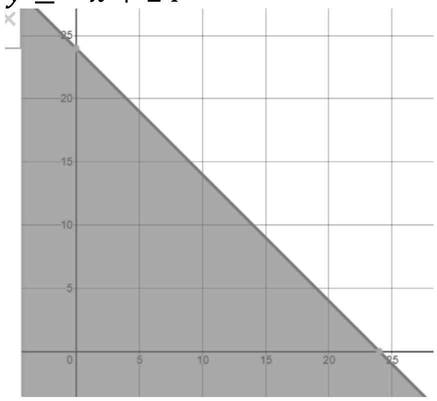
Answer Key: Optimization Unit

2) Let x be the number of strawberry baskets

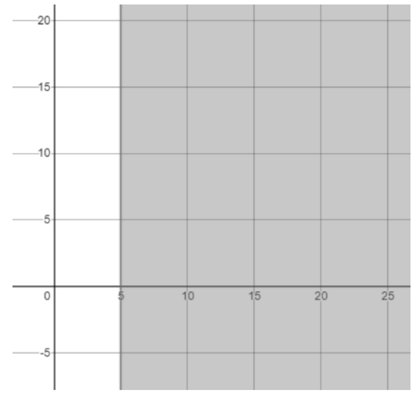
Let y be the number of flower baskets

a) $x + y \leq 24$

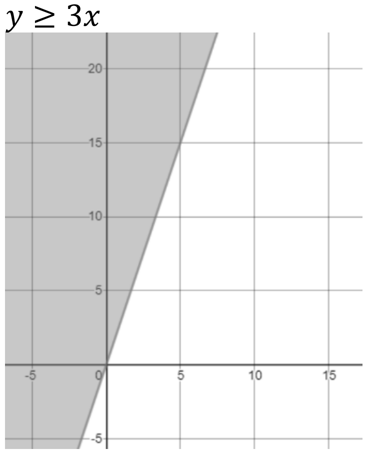
$y \leq -x + 24$



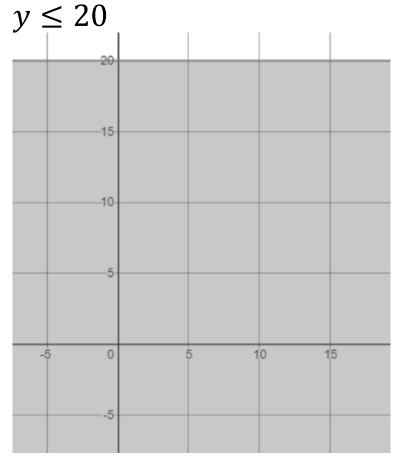
b) $x \geq 5$



c) $y \geq 3x$

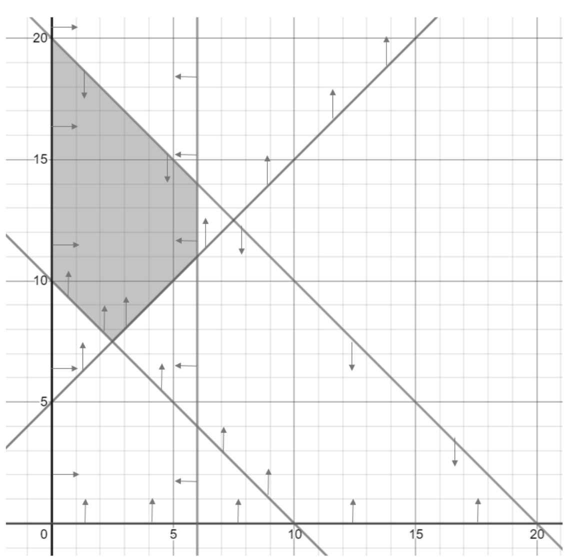


d) $y \leq 20$

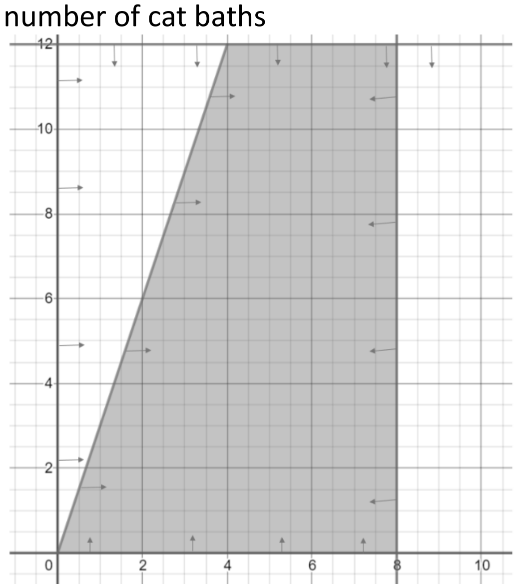


1.6 OPTIMIZATION STEP 5 ANSWERS

1) Let x be the number of stuffed bears and L=let y be the number of stuffed dinosaurs

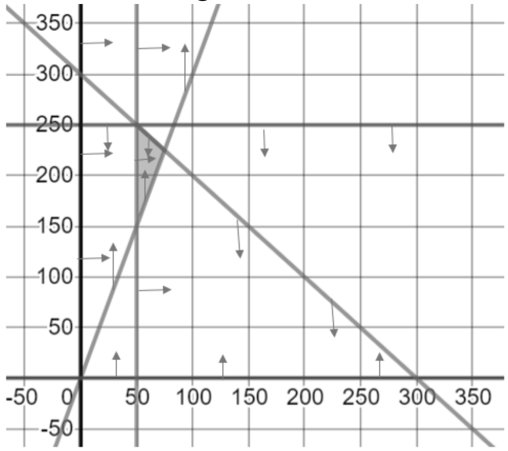


2) Let x be the number of dog baths and let y be the number of cat baths



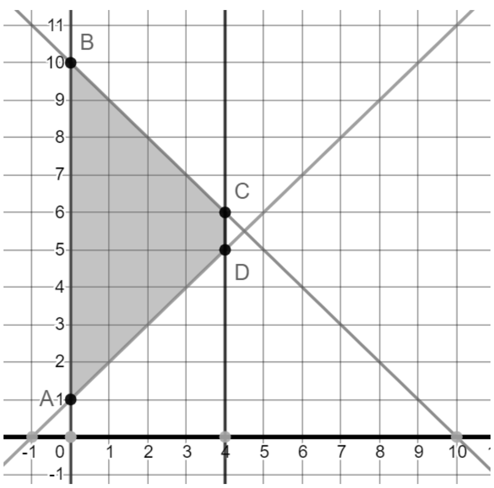
Answer Key: Optimization Unit

3) Let x be the number of donor seats and let y be the number of general admission seats



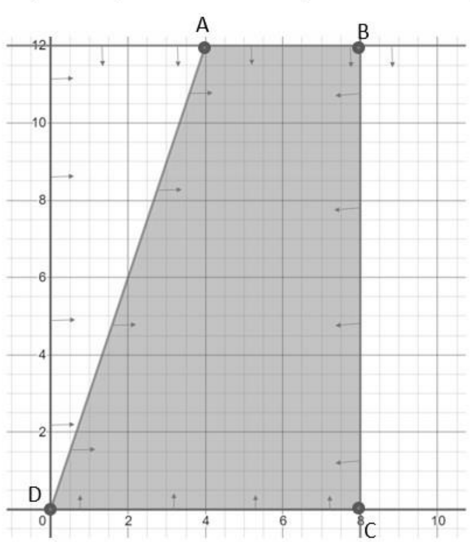
1.7 OPTIMIZATION STEP 6 ANSWERS

1) Let x be the number of stuffed bears and let y be the number of stuffed dinosaurs



Vertex A	**(0, 1)**
Vertex B	**(0, 10)**
Vertex C	**(4, 6)**
Vertex D	**(4, 5)**

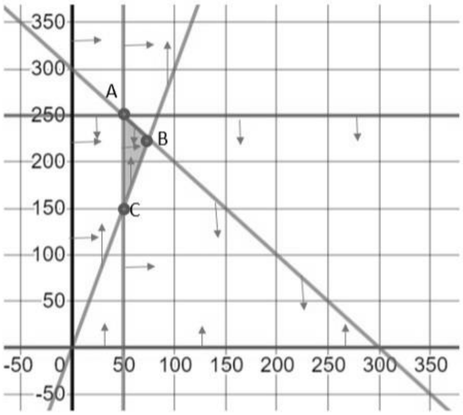
2) Let x be the number of dog baths and let y be the number of cat baths



Vertex A	**(4, 12)**
Vertex B	**(8, 12)**
Vertex C	**(8, 0)**
Vertex D	**(0,0)**

Answer Key: Optimization Unit

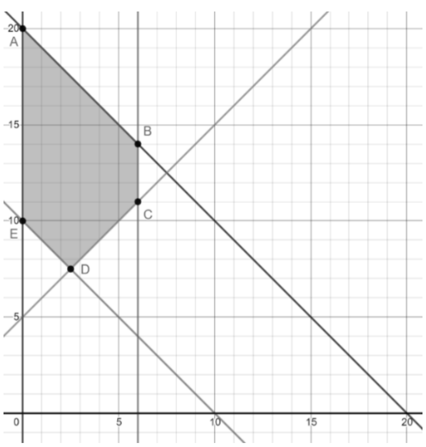
3) Let x be the number of donor seats and let y be the number of general admission seats



Vertex A	**(50, 250)**
Vertex B	**(75, 225)**
Vertex C	**(50, 150)**

1.8 OPTIMIZATION STEPS 7-10 ANSWERS

1) Let x be the number of stuffed bears and let y be the number of stuffed dinosaurs



Optimizing Function: $Value = 10x + 5y$

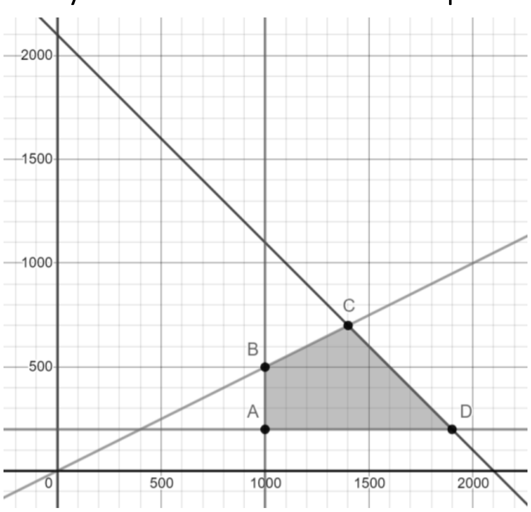
Vertex A	**(0, 20)**	$\boldsymbol{Value = 100}$
Vertex B	**(6, 14)**	$\boldsymbol{Value = 130}$
Vertex C	**(6, 11)**	$\boldsymbol{Value = 115}$
Vertex D	**(2.5, 7.5)**	$\boldsymbol{Value = 62.5}$
Vertex E	**(0, 10)**	$\boldsymbol{Value = 50}$

The minimum value of John's collection is $50 and if John's collection was worth the minimum, he would have 0 bears and 10 dinosaurs.

2)

Let x be the number of compact cars produced each week

Let y be the number of minivans produced each week



$Profit = 4000x + 10000y$

Vertex A	**(1000, 200)**	$\boldsymbol{P = 6\,000\,000}$
Vertex B	**(1000, 500)**	$\boldsymbol{P = 9000\,000}$
Vertex C	**(1400,700)**	$\boldsymbol{P = 12\,600\,000}$
Vertex D	**(1900, 200)**	$\boldsymbol{P = 9\,600\,000}$

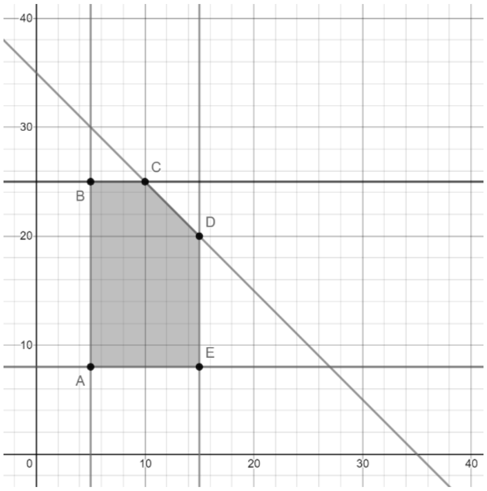
The maximum profit the manufacturer can earn weekly is $12 600 000.

Answer Key: Optimization Unit

3)

Let x be the amount of medication A

Let y be the amount of medication B



$Efficacy = 0.0305x + 0.025y$

The doctor should give 15 mg of medication A and 20 mg of medication B in order to achieve maximum efficacy.

Vertex A	**(5, 8)**	$\boldsymbol{E = 0.3525}$
Vertex B	**(5, 25)**	$\boldsymbol{E = 0.7775}$
Vertex C	**(10, 25)**	$\boldsymbol{E = 0.93}$
Vertex D	**(15, 20)**	$\boldsymbol{E = 0.9575}$
Vertex E	**(15, 8)**	$\boldsymbol{E = 0.6575}$

1.9 OPTIMIZATION COMPLICATIONS: DECIMALS, DOTTED LINES, and TIES ANSWERS

1) $Value = 2x + 1.5y$

Vertex A	**(1, 2)**	$Value = 5$
Vertex B	**(1, 9)**	$Value = 15.5$
Vertex C	**(3.33, 6.66) change to (3, 7)**	$Value = 16.5$

The maximum value of Reese's collection is $16.50.

2) $Profit = 4x + 1.5y$

Vertex A	**(5, 5)**	$\boldsymbol{Profit = 27.5}$
Vertex B	**(5, 20)**	$\boldsymbol{Profit = 50}$
Vertex C	(15, 10) change to **(14, 11)**	$\boldsymbol{Profit = 72.5}$
Vertex D	(15, 5) change to **(14, 6)**	$\boldsymbol{Profit = 67.5}$

The pizza shop must sell 11 slices of pizza to earn the maximum profit.

Answer Key: Optimization Unit

3) $Profit = 3x + 6y$

Vertex A (2, 3)
$Profit = 24$

Vertex B (3, 6)
$Profit = 45$

Vertex C (9, 3)
$Profit = 45$

Vertex D (6, 1)
$Profit = 24$

Vertices B and C both tie for the maximum profit. Therefore, they are both solutions, as are all the points in between them.

There are 4 points that maximize the situation: (3, 6), (5, 5), (7, 4), and (9, 3).

4)

Vertex A (0, 2)
$Cost = 4$

Vertex B (0.8, 3.6)
Change vertex to (1, 3)
$Cost = 10$

Vertex C (4, 2)
$Cost = 20$

Minimum cost is $4.

5)

Vertex A (0, 4)
$Value = 40$

Vertex B (2, 8)
Change to (2, 7)
$Value = 74$

Vertex C (6, 6)
Change to (5, 6)
$Value = 70$

Maximum Value is $74.

6)

Vertex A (2, 8)
$Value = 72$

Vertex B (8, 5)
$Value = 72$

Vertex C (4, 0)
$Value = 16$

Vertex D (2, 0)
$Value = 8$

Maximum Value is $72.
Both vertices A and B produce this maximum value, so all the points on the line connecting A and B are also solutions. There are 4 points that maximize the scenario.

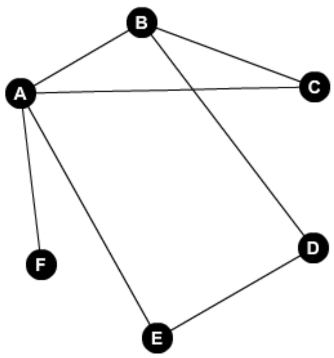
Answer Key: Optimization Unit
1.10 OPTIMIZATION EXAM STYLE QUESTIONS ANSWERS

1) C
2) B
3) C
4) (5, 1)
5) $x + y \leq 200$ $(or\ y \leq -x + 200)$ and $x \geq 2y$ $(or\ 2y \leq x\ or\ y \leq \frac{1}{2}x\ or\ y \leq 0.5x)$
6) The expected maximum revenue will decrease by $500.
7) The maximum Mackenzie can earn each day is $80.
8) The profit earned on each bottle of body lotion is $2.20.
9) $650 000

Answer Key: Graph Theory Unit

2.1 GRAPH THEORY INTRODUCTION

1) (the graph below is one example of a correct answer)



2)

Vertices (and degree of each): A – degree 2, B – degree 2, C – degree 3, D – degree 3, E – degree 2

Edges: AB, AC, BD, CD, CE, DE

Order: 5

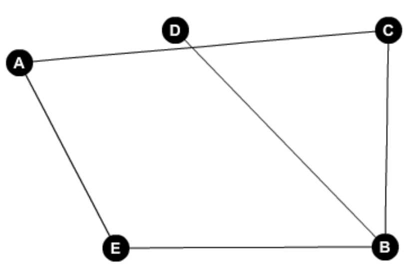
3)

Loop(s): EE

Parallel Edge(s): CD(1) and CD(2); BD(1) and BD(2)

4a) Neither 3b) Connected 3c) Complete (and connected)

5)



Answer Key: Graph Theory Unit

6)

a) A – 2; B – 3; C – 3; D – 2; E – 2; F – 2

b) AB, AD, BE(1), BE(2), CC, CF, DF

c) 6

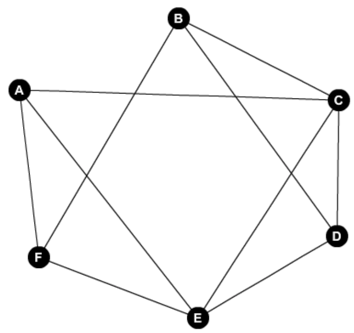
d) CC

e) BE(1), BE(2)

f) yes

g) no

h)



7)

a) A – 1; B – 4; C – 4; D – 2; E – 1; F – 2

b) AE, BC(1), BC(2), BD, BF, CD, CF

c) 6

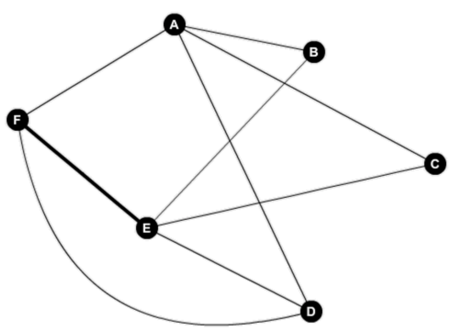
d) none

e) BC(1), BC(2)

f) no

g) no

h)



2.2 GRAPH THEORY PATHS AND CIRCUITS

1)

a) many possible solutions

b) many possible solutions

c) length is 6

d) many possible solutions

e) $d(C, A) = 2$

f) $d\ (A, D) = 1$

2)

a) multiple options

b) multiple options

c) multiple options

d) multiple options

e) $d(A, C) = 2$

Answer Key: Graph Theory Unit

2.3 EULER PATH AND CIRCUIT

1)

a) Yes, an Euler path exists
Many options

b) No, an Euler path does not exist.

2)

a) Yes, an Euler circuit does exist.
Many options

b) Yes, an Euler circuit does exist.
Many options

2.4 HAMILTONIAN PATH AND CIRCUIT

1) Many options

2) Many options

3)

a) Euler path, Hamiltonian path

b) Hamiltonian path, Hamiltonian circuit

c) Hamiltonian path, Hamiltonian circuit

d) Euler path, Euler circuit, Hamiltonian path

2.5 DIRECTED GRAPHS

1) a) Many options b) Many options c) no (cannot go from A to E)
d) 5 e) 2

2.6 WEIGHTED GRAPHS (NETWORKS)

1) a) 10 b) 26

2) a) 13 b) 13
c) 20 d) E-D-C-A-B; E-A-D-C-A-B; E-A-B e) E-D-C-A-B

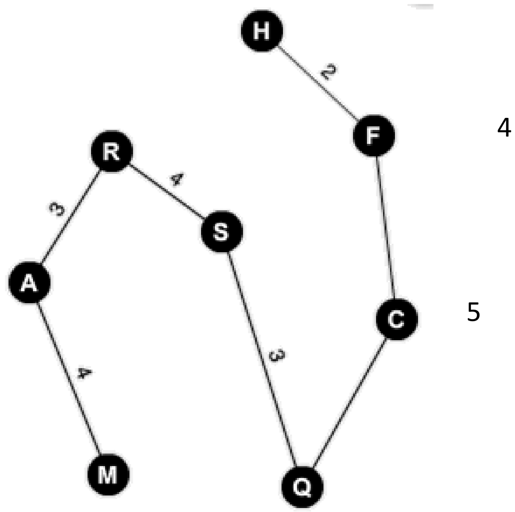
3) a) Many options b) 29 km
c) Many options

2.7 PATH OF OPTIMAL VALUE

1) a) 80 b) 35

Answer Key: Graph Theory Unit
2.8 TREE OF OPTIMAL VALUE

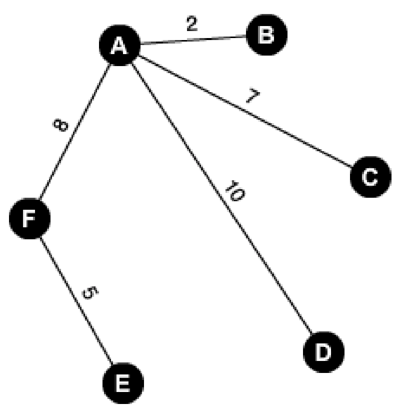
1)



or same graph but not QC and add SF

Total power loss is 25 (or 250 megawatts)

2)



32 km

3) a) remove one of the following: AC, BC, BD, or AD
   b) add one of the following: AD, AC, AB, ED, EC, or EB
   c) add one of the following: AC, AD, BC, or BD
   d) remove one of the following: AB, AE, or BE

4) $62 000.00

5) a) $25.50
   b) $28.50

Answer Key: Graph Theory Unit

2.9 CRITICAL PATH

1) 66 days

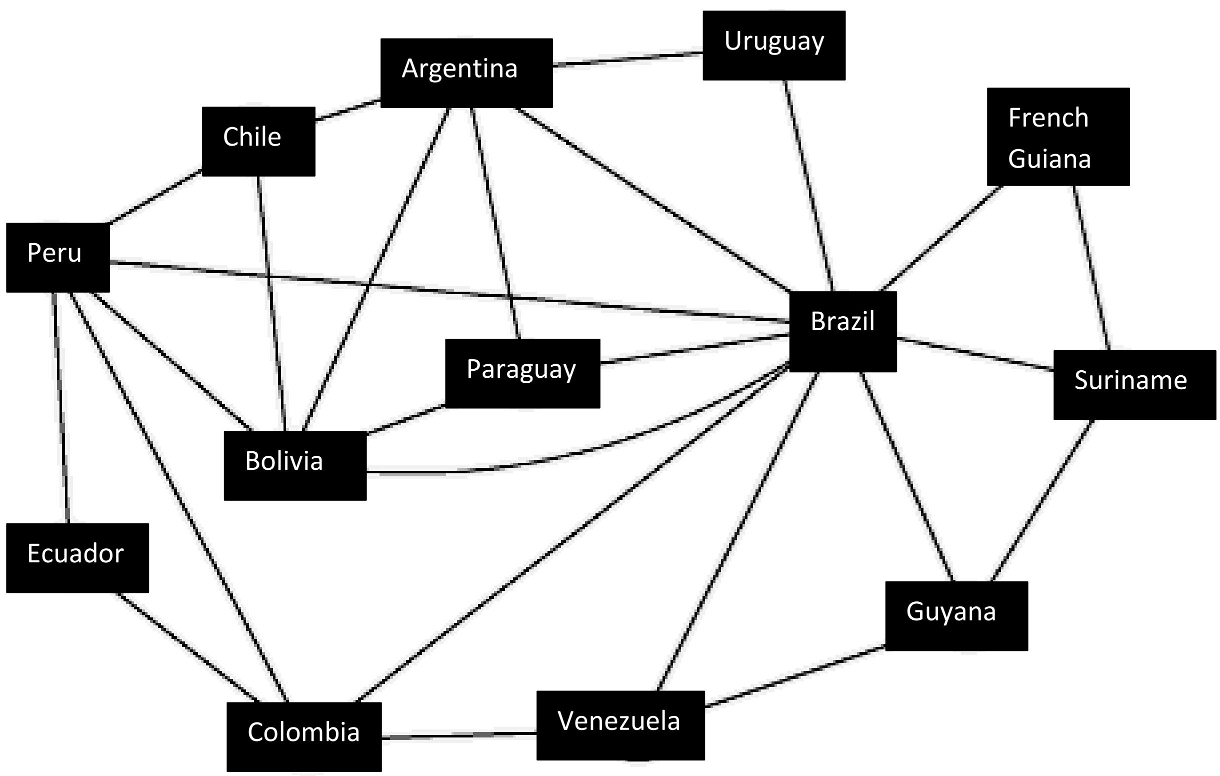
2) He does not have enough time (he needs 80 minutes but only has 75)

3) 13 days

2.10 CHROMATIC NUMBER

1) a) 3 b) 4 c) 3

2) a)



b) 4 c) yes

Answer Key: Graph Theory Unit
2.11 GRAPH THEORY EXAM STYLE QUESTIONS

1) C – Hamiltonian Path

2) C – 13.1 km

3) B – $660

4) D

5) B – 3

6) 3 groups: Amanda and Eli; Bob and Chris; David and Francis

7) 130 minutes

8) Option 2. It will save the company 3 days.

Answer Key: Financial Math Unit

3.1 EXPONENTS REVIEW

1) a) 1,728 b) 0.188

c) 1,280 d) 101.91

3.2 POPULATION AND COMPOUND INTEREST

1) 170 students will attend

2) There will be 207 species (206.6)

3) There will be 16,384 bacteria

4) $6,691.13

5) $222.23

6) $2,865.46

3.3 SOLVING FOR OTHER VARIABLES

1) 50% per day

2) 12% per hour

3) 389.7 cm

4) $6000.00

5) $3000.03

6) 8% per year

7) 5% per year

8) 7% per year

3.4 LOGARITHM BASICS

1) a) 2.73 b) 0.64 c) 162.11

Answer Key: Financial Math Unit

3.5 USING LOGS TO SOLVE POPULATION AND FINANCIAL MATH QUESTIONS

1) 6 years

2) 5 years

3) 6.6 hours

4) 10.5 hours

3.6 SIMPLE INTEREST

1) $10,800

2) $4960.80

3) Investment 1: $3298.87 Investment 2: $1761.60 Total: $5060.47

Yes, Mikka will have enough money.

3.7 SOLVING FOR OTHER VARIABLES

1) $2675

2) 40 months

3) 20% monthly simple interest rate

4) 1460 days

5) $10,462.89

6) 2.85% quarterly simple interest rate

Answer Key: Financial Math Unit

3.8 POPULATION AND FINANCIAL MATH EXAM STYLE QUESTIONS

1) D

2) $1,200

3) Team A: $2286.20 Team B: $3713.94 Total: $6000.14

It will take Team C 7.5 years

4) Car 1: $7986.70 Car 2: $7788.32 She will buy Car 2

She invested her money 5.9 years ago.

Answer Key: Voting Unit

4.1 Plurality

1) Candidate B

2) Tacos

3) Village A

4.2 Majority

1) No Winner

2) Montreal

4.3 Majority with Elimination

1) C

2) Blue

4.4 Borda Count

1) Movie

2) Math

4.5 Condorcet Method

1) Blue

2) No Winner

4.6 Proportional Representation

1) a) Liberal: 114 seats; Conservative: 119 seats; BQ: 26 seats; NDP: 55 seats; Green: 23 seats; Independent: 1 seat

   b) Liberals would lose 43 seats; Conservatives would lose 2 seats; BQ would lose 6 seats; NDP would gain 31 seats; Green would gain 20 seats; Independents would stay the same

Answer Key: Voting Unit
4.7 Approval

1) Vietnamese

2) Tacos

4.8 Exam Style Questions

1) Boston

2) Erin, Brandon, and Charlie

3) The teachers are incorrect in their prediction. Media wins twice (plurality and Condorcet) and Biology wins twice (elimination and Borda).

Answer Key: Probability Unit

5.1 Review

1) $P(O) = \frac{15}{100}\ or\ \frac{3}{20}$

2) a) $P(Q) = \frac{4}{52}\ or\ \frac{1}{13}$ b) $P(R) = \frac{26}{52}\ or\ \frac{1}{2}$ c) $P(7d) = \frac{1}{52}$

3) a) $P(R) = \frac{4}{15}$ b) $P(G) = \frac{2}{15}$ c) $P(W) = \frac{6}{15}$ d) $P(B) = \frac{3}{15}\ or\ \frac{1}{5}$

5.2 Probabilities with "And" or "Or"

1) $P(4 \cup Q) = \frac{8}{52}\ or\ \frac{2}{13}$

2) $P(7 \cup R) = \frac{28}{52}\ or\ \frac{7}{13}$

3) $P(4 \cap 6) = \frac{2}{36}\ or\ \frac{1}{18}$

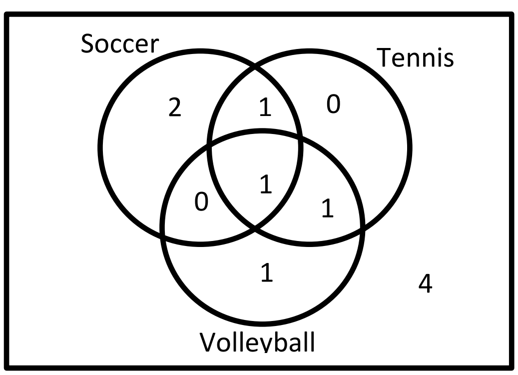
4) $P(4 \cup 6) = \frac{20}{36}\ or\ \frac{5}{9}$

5) $P(B \cap H) = \frac{4}{14}\ or\ \frac{2}{7}$

5.3 Venn Diagrams

1) a)

$\Omega = 10$

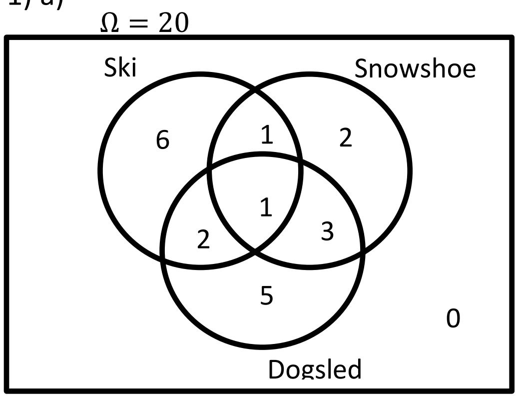


2) a) $P(A) = \frac{39}{78}\ or\ \frac{1}{2}$ b) $P(A \cap B) = \frac{16}{78}\ or\ \frac{8}{39}$ c) $P(B \cup C) = \frac{60}{78}\ or\ \frac{10}{13}$ d) $P(A \cap B \cap B) = \frac{12}{78}\ or\ \frac{2}{13}$

Answer Key: Probability Unit

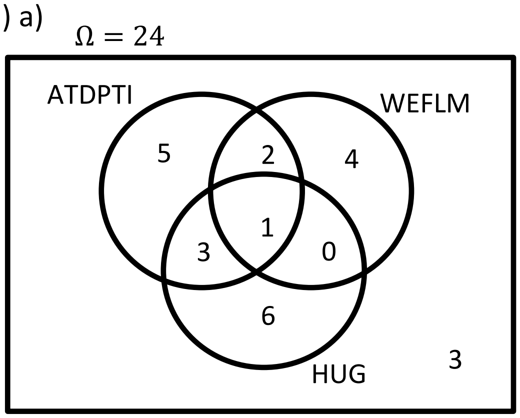
5.4 Constructing Venn Diagrams

1) a)



b) $P(Dogsled \cup Ski) = \frac{18}{20}$ $or\ \frac{9}{10}$

2) a)



b) $P(none) = \frac{3}{24}$ $or\ \frac{1}{8}$

5.5 Contingency Tables

1) a) $P(S \cap C) = \frac{12}{115}$ b) $P(R) = \frac{23}{115}$ $or\ \frac{1}{5}$ c) $P(P \cup Chips) = \frac{85}{115}$ $or\ \frac{17}{23}$

2) a)

	Home Economics	Biology	Law	Total
Art	15	25	80	120
Music	25	15	10	50
Drama	20	10	0	30
Total	60	50	90	200

b) $P(D \cap HE) = \frac{20}{200}$ $or\ \frac{1}{10}$

Answer Key: Probability Unit

5.6 Conditional Probability

1) a) $P(B|A) = \frac{12}{30}\ or\ \frac{2}{5}$ b) $P(A|B) = \frac{12}{20}\ or \frac{3}{5}$

2) a) $P(S|C) = \frac{12}{30}\ or\ \frac{2}{5}$ b) $P(P|R) = \frac{12}{23}$ c) $P(M|Chips) = \frac{17}{41}$ d) $P(C|T) = \frac{7}{29}$

5.7 Random Experiment with Several Steps

1) a) $P(one\ of\ each) = \frac{9}{50}$ b) $P(2\ red, 1\ black) = 1/6$

c) $P(3\ of\ a\ kind) = \frac{67}{750}$ d)$P(2\ black, 1\ yellow) = \frac{9}{250}$

5.8 Permutations and Combinations

1) 4096

2) 3,603,600

3) a) 35 b) $P(GGG) = \frac{1}{35}$ c) $P(3\ of\ a\ kind) = \frac{5}{35}$

4) a) 90 b) $P(consecutive\ numbers) = \frac{9}{90}\ or \frac{1}{10}$

5.9 Odds

1) a) $\frac{2}{5}\ or\ 2{:}\,5$ b) $\frac{5}{2}\ or\ 5{:}\,2$

2) a) $\frac{6}{30}\ or\ 6{:}\,30\ or \frac{1}{5}\ or\ 1{:}\,5$ b) $\frac{35}{1}\ or\ 35{:}\,1$

c) $\frac{1}{35}\ or\ 1{:}\,35$ d) 0

5.10 Expectation and Fairness

1) $Expected\ Gain = \$-0.40$

2) $Expected\ Gain = \$1.50$

3) No

4) a) No b) $2.22

Answer Key: Probability Unit
5.11 Weighted Mean

1) 71.9%

2) Car A

5.12 Probability Exam Style Questions

1) A

2) A

3) B

4) a) 12 b) 30

5) a) 2248 b) $\frac{1686}{2764}$ or $\frac{843}{1382}$

6) $\frac{15}{65}$ or $\frac{3}{13}$

7) Die Roll

8) $P(NYC|Grade\ 11) = \frac{60}{250}$ or $\frac{6}{25}$

9) The Fruit Wheel is to Serena’s disadvantage. The expected gain is $\$-0.67$

Answer Key: Geometry Unit

6.1 Area and Perimeter

a) $A = 150\ cm^2$
$P = 53.31\ cm$

b) $A = 28\ cm^2$
$P = 22\ cm$

c) $A = 96\ m^2$
$P = 40\ m$

d) $A = 482.8cm^2$
$P = 80\ cm$

e) $A = 153.94\ m^2$
$P = 43.98\ cm$

f) $A = 25\ km^2$
$P = 20\ cm$

6.2 Triangles

1)

a) 7.21 units

b) 110.41 units

c) 1.55 units

2)

a) 19.42 cm

b) 24.25 cm

3)

a) 5 cm^2

b) 1.95

4)

a) side: 3.85 cm
angle: 36°
angle: 95°

b) angle: 38.4°
angle: 30.3°
angle: 111.3°

c) angle: 56.4°
angle: 60.1°
angle: 63.5°

d) side: 17.28 cm
angle: 40.3°
angle: 27.3°

6.3 Finding the Apothem

1)

a) 6.9 m

b) 1.85 m

c) 103.8 cm

d) 5.6 m

6.4 Missing Measures

1)

a)
$r = 3.97\ ft$
$d = 7.94\ ft$
$A = 49.51\ ft^2$

b)
$r = 10.3\ in$
$d = 20.6\ in$
$A = 64.7\ in$

c)
$b = 8\ m$
$p = 33.3\ m$

d)
$h = 22\ m$
$p = 63\ m$

e)
$b = 8ft$

f)
$h = 10\ yd$

Answer Key: Geometry Unit

6.5 Equivalent Figures

1) $s = 6\ cm$

2)

a) 9.8 cm

b) 3 cm

c) 8 cm

d) 5 cm

6.6 Rules of Equivalency

1) Equilateral triangle

2) Hexagon

3) Hexagon

4) 16 cm^2

5) 6.92 cm^2

6.7 Lateral Area, Surface Area, and Volume

1)

a)

$$A_l = 125.7\ units^2$$
$$A_s = 150.8\ units^2$$
$$V = 125.7\ units^3$$

b)

$$A_l = 630\ units^2$$
$$A_s = 790\ units^2$$
$$V = 1200\ units^3$$

c)

$$A_l = 96\ km^2$$
$$A_s = 144\ km^2$$
$$V = 96\ km^3$$

d)

$$A_l = 1040\ units^2$$
$$A_s = 1440\ units^2$$
$$V = 3200\ units^3$$

e)

$$A_l = 1176.2\ units^2$$
$$A_s = 1377.3\ units^2$$
$$V = 1473.8\ units^3$$

6.8 Missing Measures

1) $b = 6\ ft$

2) $h = 4\ cm$

3) 2mm

Answer Key: Geometry Unit

6.9 Equivalent Solids

1) Yes

2) 6 cm

3) 7.2 cm

6.10 Rules of Equivalency

1) 125 cm^3

2) 384 cm^2

3) 113 cm^3

6.11 Geometry Exam Style Questions

1) B

2) C

3) D

4) 8.9cm

5) sphere, dodecahedron, octahedron, cube

6) No. Old edging was 22 sm. New edging is 66cm

7) $48,080