Voting Unit – Exam Style Questions

Probability Unit – Review
5.1 REVIEW

Recall that there are several types of probability: theoretical, experimental, and subjective.

- The **theoretical probability** of an event is a number that quantifies the possibility that the event will occur – it is determined by mathematical reasoning.

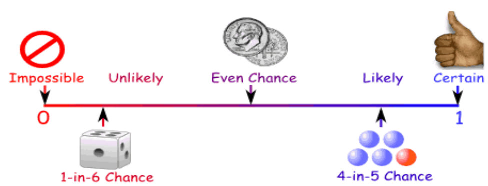
$$Theoretical\ Probability = \frac{\#\ of\ favorable\ outcomes}{\#\ of\ possible\ outcomes}$$

- The **experimental probability** of an event is determined from the experiment itself.

$$Experimental\ Probability = \frac{\#\ of\ times\ outcomes\ occurs}{\#\ of\ trials}$$

- The **subjective probability** that an event will occur relies on judgment or perceptiveness of a person who has a particular set of information about the situation.

Probability is the chance something will happen. This can be shown on a probability line.

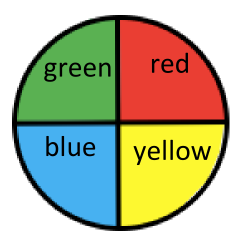


We can also use numbers (such as fractions or decimals) to show the probability of something happening.

- Impossible is a zero
- Certain is a one

We can calculate the probability of an event using the following formula:

Probability of Event A: P(A)
$P(A) = \frac{Number\ of\ ways\ A\ occurs}{Total\ number\ of\ ways}$



Ex: A spinner has 4 equal sectors coloured yellow, blue, green and red.

What are the chances of landing on blue? $P(B) = \frac{1}{4}$

What are the chances of landing on red? $P(R) = \frac{1}{4}$

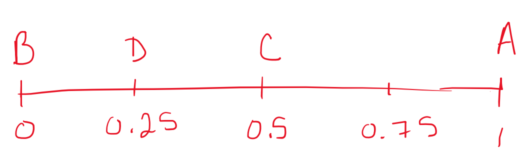
Ex: Where would the following statements sit on the probability line?

A) Choosing a green ball from a bag with only 4 green balls

B) Choosing a green ball from a bag with only 2 red balls and 4 yellow balls

C) Choosing a green ball from a bag with 2 green balls and 2 red balls

D) Choosing a red ball from a bag with 1 red ball and 3 green balls



Probability Unit – Review

Practice Questions

1) A jar contains 100 jelly beans. There are 20 red jelly beans, 10 black jelly beans, 15 orange jelly beans, 30 green jelly beans, and 25 yellow jelly beans. If you chose a single jelly bean at random, what is the probability that you would choose an orange jelly bean?

2) In a standard deck of 52 playing cards, what is the probability of randomly drawing:

a) a queen?

b) a red card?

c) a club?

d) the 7 of diamonds?

3) A bag contains 4 red marbles, 2 green marbles, 6 white marbles, and 3 blue marbles. What is the probability of randomly drawing:

a) a red marble?

b) a green marble?

c) a white marble?

d) a blue marble?

Probability Unit – Probabilities with "And" or "Or"

5.2 PROBABILITIES WITH "AND" OR "OR"

The probability of A and B means we want to know the probability of two events happening at the same time.
For now, we will only consider **independent events**. This means the probability of one event does not change the probability of another event.

Consider the following: you flip a coin and roll a die. The result of the coin flip does not alter the roll of the die – these are **independent events**.

Now consider drawing two cards from a deck of 52 cards. After you draw the first card, there are fewer cards to draw for the second card, thus changing the probabilities. These are **dependent events**, and we will address them later.

To determine the probability of two independent events occurring, multiply probabilities: $$P(A \cap B) = P(A) \times P(B)$$

Ex: Dr. James draws one card from a deck of 52 cards and flips a coin once. What is the probability she drew a 5 and the coin landed on tails?

$P(5 \cap T) = P(5) \times P(T)$

$= \frac{1}{6} \times \frac{1}{2}$

$P(5 \cap T) = \frac{1}{12}$

Ex: You roll two dice. What is the probability you roll a 6 on both?

$P(6 \cap 6) = P(6) \times P(6)$

$= \frac{1}{6} \times \frac{1}{6}$

$P(6 \cap 6) = \frac{1}{36}$

Ex: You roll two dice. What is the probability you roll a 4 on one die and a 2 on the other?

SKIP

Probability Unit – Probabilities with “And” or “Or”

The probability of A or B means we want to know the probability of either of the two events happening.
We will continue to consider only independent events.

To determine the probability either of two independent events occurring: $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex: Dr. James draws one card from a deck of 52 cards and flips a coin once. What is the probability she either drew a 5 or the coin landed on tails?

$$P(5 \cup T) = P(5) + P(T) - P(5 \cap T)$$
$$= \frac{4}{52} + \frac{1}{2} - \left(\frac{4}{52}\right) \times \left(\frac{1}{2}\right)$$
$$= \frac{15}{26} - \frac{1}{26}$$
$$P(5 \cup T) = \frac{7}{13} \text{ or } 0.5385$$

Ex: You roll a green die and a blue die. What is the probability you roll a 3 on the green die or a 5 on the blue die?

$$P(3_g \cup 5_b) = P(3_g) + P(5_b) - P(3_g \cap 5_b)$$
$$= \frac{1}{6} + \frac{1}{6} - \left(\frac{1}{6}\right) \times \left(\frac{1}{6}\right)$$
$$= \frac{1}{3} - \frac{1}{36}$$
$$P(3_g \cup 5_b) = \frac{11}{36} \text{ or } 0.3056$$

Ex: You roll two dice. What is the probability you roll a 4 on one die and a 2 on the other?

SKIP

Probability Unit – Probabilities with “And” or “Or”

Practice Questions

1) You draw one card from a standard deck of cards. What is the probability you draw a 4 or a queen?

2) You draw one card from a standard deck of cards. What is the probability you draw a 7 or a red card?

3) You roll two dice. What is the probability you roll a 4 and a 6?

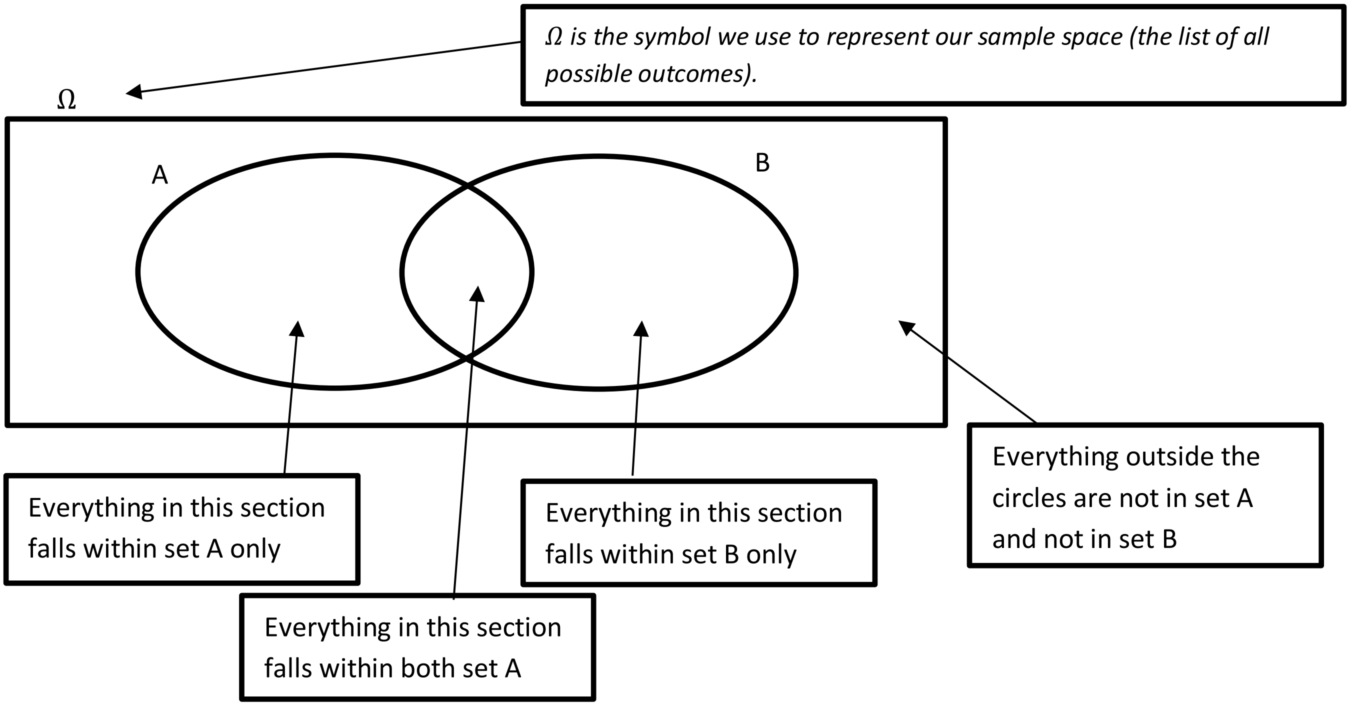
4) You roll two dice. What is the probability you roll a 4 or a 6?

5) You draw a marble from a bag containing 4 blue marbles and 3 red marbles. You also flip a coin. What is the probability you draw a blue marble and the coin lands on heads?

Probability Unit – Venn Diagrams

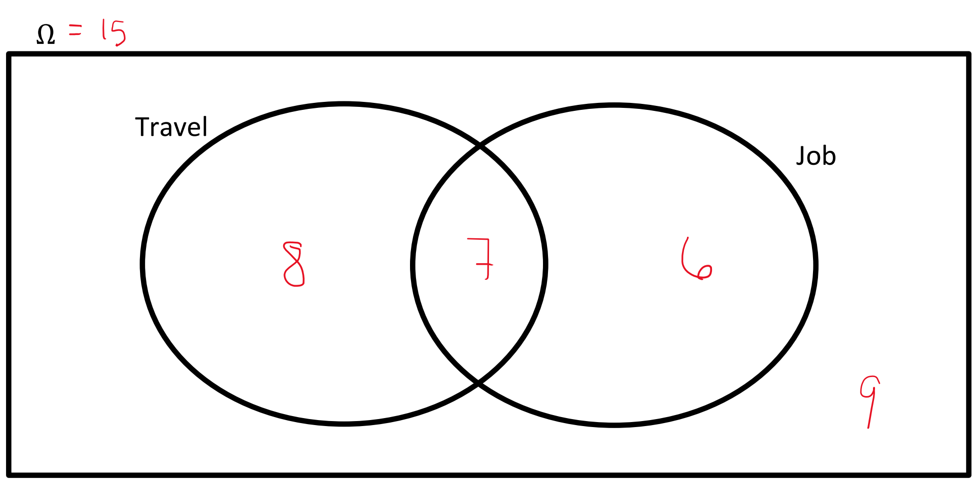
5.3 VENN DIAGRAMS

Venn diagrams are great because we can use them to think about probability for independent of dependent events. We won't even have to think about whether the events are independent or dependent.



Ex: Dr. James asked 30 students the following questions: 1) Do you travel more than 30 minutes to get to school? 2) Do you have a job? Use the information in the table below to create a Venn diagram.

Yes to Q1 only (travel)	**Yes to Q2 only (job)**	**Yes to both Q1 and Q2**	**No to both**
8 students	6 students	7 students	9 students

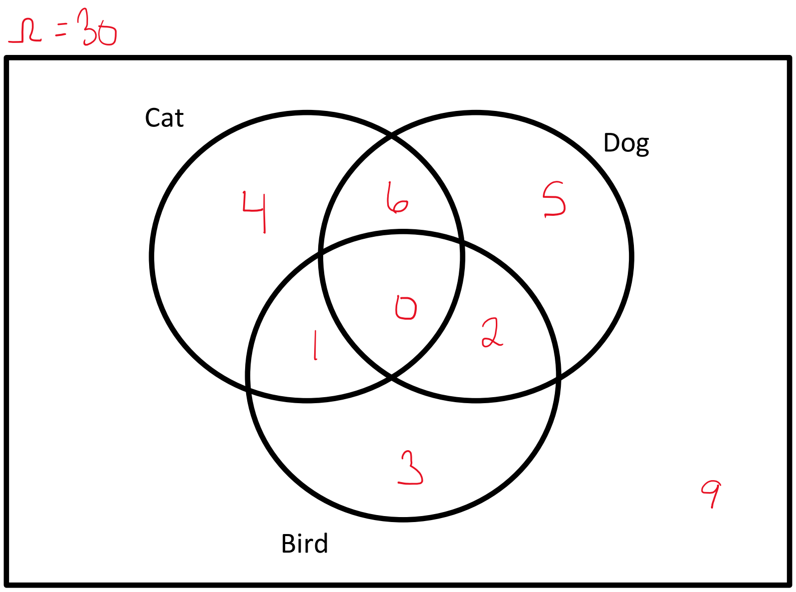


Probability Unit – Venn Diagrams

We can also create Venn Diagrams with more than two circles.

Ex: Dr. James asked 30 students the following questions: 1) Do you own a cat? 2) Do you own a dog? 3) Do you own a bird? The results of the survey are presented in the table below and depicted in the Venn Diagram. Use the information in the table below to create a Venn diagram.

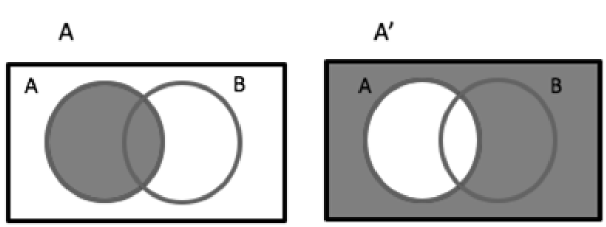
Yes to Cat only	**Yes to Dog only**	**Yes to Bird only**	**Yes to Cat and Dog**	**Yes to Cat and Bird**	**Yes to Dog and Bird**	**Yes to Cat, Bird, and Dog**	**No to all**
4 students	5 students	3 students	6 students	1 student	2 students	0	9 students



Venn diagrams are especially useful for determining "AND" and "OR" probabilities. They also let us visualize the complement of an event. The **complement** of an event is the collection of outcomes that are NOT the event.
In the spinner example, if the event (A) is spinning the spinner and landing on BLUE, the complementary event (A') is landing on NOT BLUE (everything other than blue).

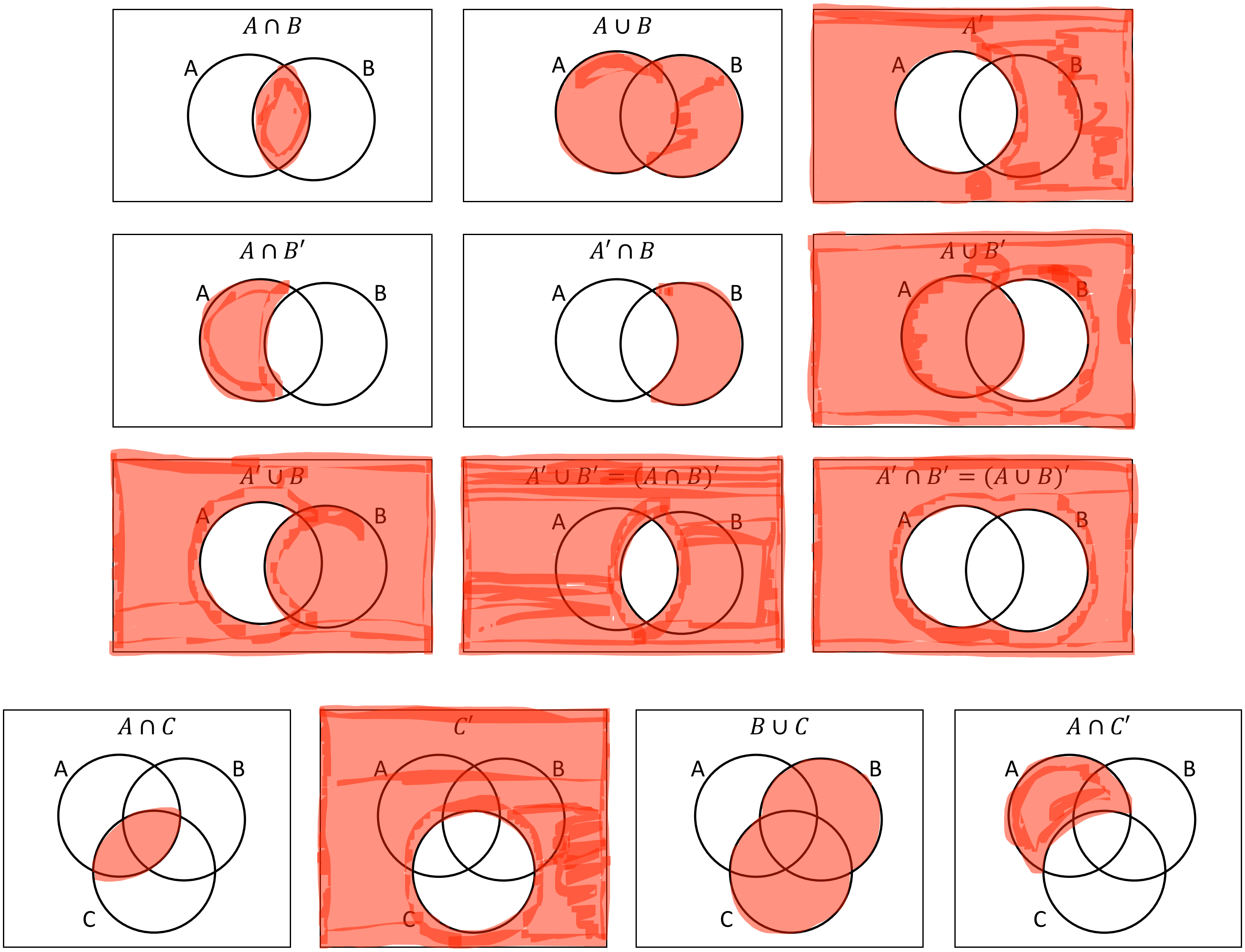
When you add the probability of an event, (A) to the probability of the complement of the event (A'), the result will always be 1. It is sometimes easier to determine P(A) using this method:

$$P(A) + P(A') = 1$$



Probability Unit – Venn Diagrams

Ex: Shade the appropriate section of each Venn diagram.

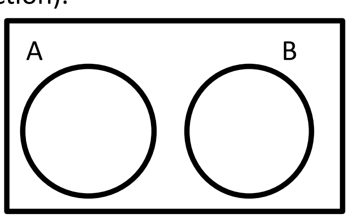


Events are called **mutually exclusive** if they cannot happen at the same time.

Ex: If we are spinning a spinner one time and Event A=spinning BLUE and Event B=spinning RED, those events are mutually exclusive because you cannot land on BLUE and RED at the same time.

In a Venn Diagram, two mutually exclusive events do not overlap (or if the circles are drawn as overlapping, there is nothing in that overlapping section).

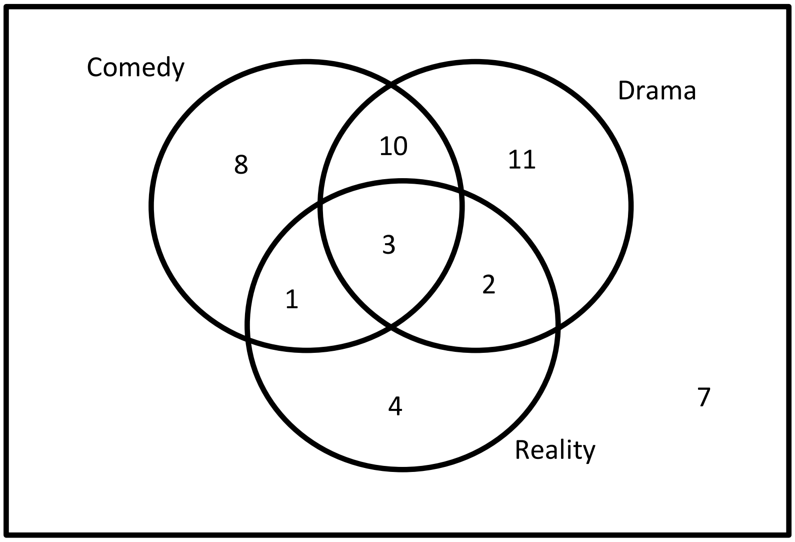
$P(A \cap B) = 0$



Probability Unit – Venn Diagrams

Ex: A group of ~~100~~ people were asked what types of television shows they watched. The results are presented in the Venn diagram below.

$\Omega = 46$



Determine the following probabilities:

- A person watches comedy shows $P(C) = \frac{8+10+1+3}{46} = \frac{22}{46} = \frac{11}{23}$ or 0.4783
- A person watches drama or reality shows $P(D \cup R) = \frac{10+11+3+2+1+4}{46} = \frac{31}{46}$ or 0.6739
- A person watches comedy and reality shows $P(C \cap R) = \frac{1+3}{46} = \frac{4}{46} = \frac{2}{23}$ or 0.0870
- A person watches comedy and drama shows $P(C \cap D) = \frac{13}{46}$ or 0.2826
- A person watches reality and drama shows $P(R \cap D) = \frac{5}{46}$ or 0.1087
- A person watches comedy, drama, and reality shows $P(C \cap D \cap R) = \frac{3}{46}$ or 0.0652

Practice Questions

Probability Unit – Venn Diagrams

1) There are 10 friends, Alex, Blaire, Casey, Drew, Erin, Francis, Glen, Hunter, Ira, and Jade.

Alex, Casey, Drew, and Hunter all play soccer.

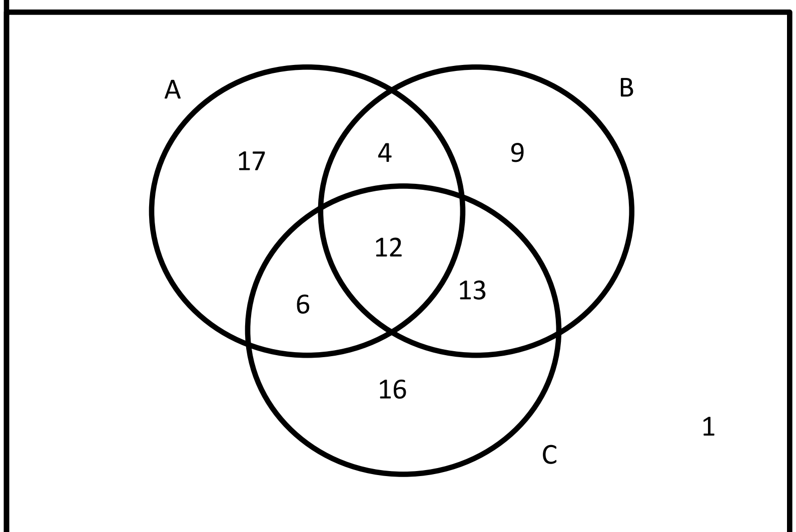
Casey, Drew, and Jade all play tennis.

Drew, Glen, and Jade all play volleyball

Create a Venn diagram to represent this scenario.

2) The Venn diagram below shows the probabilities of event A, event B, and event C.

$\Omega = 78$



Determine the following probabilities:

a) $P(A)$

b) $P(A \cap B)$

c) $P(B \cup C)$

d) $P(A \cap B \cap C)$

Probability Unit – Completing a Venn Diagram

5.4. CONSTRUCTING VENN DIAGRAMS

Often, before we can use a Venn diagram to determine probabilities, we will need to construct one given a series of clues.

Ex: 30 students were asked 3 questions: 1) Do they live on a farm; 2) Do they have a sibling; 3) Do they have a pet. Using the results below, create a Venn diagram and determine the probability that a student lives on a farm or has a sibling.

- 16 students live on a farm 6
- 17 students have a sibling 5
- 28 students have a sibling or have a pet 7
- 3 students who live on a farm own a pet and have a sibling 2
- 5 students have a sibling and own a pet 3
- 11 students live on a farm and have a sibling 4
- 26 students own a pet or live on a farm 8
- 1 student does not live on a farm, does not have a sibling, and does not have a pet 1

$16 = 8 + 3 + x + y$

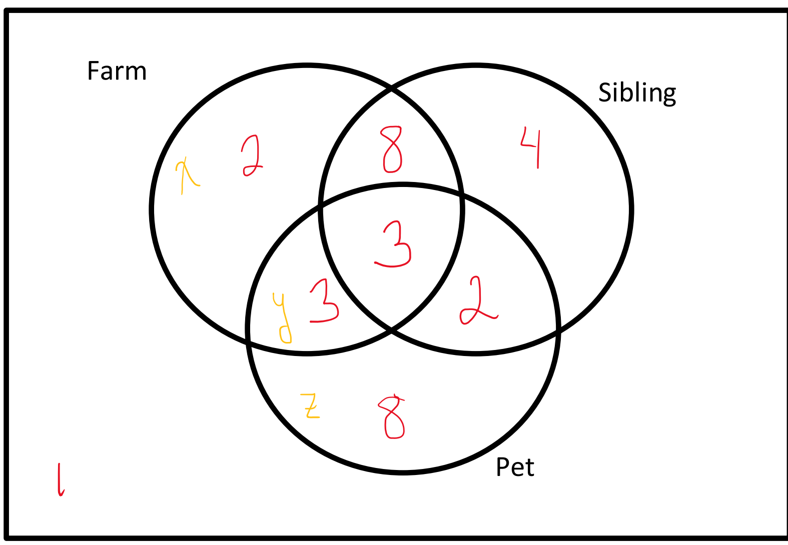
$\boxed{5 = x + y}$

$28 = 8 + 4 + 3 + 2 + y + z$

$\boxed{11 = y + z}$

$26 = 8 + 3 + 2 + x + y + z$

$\boxed{13 = x + y + z}$



$\boxed{x + y} + z = 13$

$5 + z = 13$

$z = 8$

$y + z = 11$

$y + 8 = 11$

$y = 3$

$x + y = 5$

$x + 3 = 5$

$x = 2$

Probability Unit – Completing a Venn Diagram

Practice Questions

1) 20 students went on an outdoor winter trip. The students could choose to ski, snowshoe, or dogsled. All students participated in at least one activity.

- 10 students went skiing
- 7 students went snowshoeing
- 11 students went dogsledding
- 2 students chose to both ski and snowshoe
- 4 students went snowshoeing and dogsledding
- 3 students went skiing and dogsledding.
- 1 student did all three activities

a) Complete a Venn diagram to represent this scenario.

b) If one student was chosen at random, what is the probability that the student went dogsledding or skiing?

Probability Unit – Completing a Venn Diagram

2) An English teacher took a class survey to determine what books the students had read. Of the 24 students in the class:

- 11 students had read The Absolutely True Diary of a Part-Time Indian
- 7 students had read When Everything Feels Like the Movies
- 10 students had read The Hate U Give
- 3 students had read When Everything Feels Like the Movies and The Absolutely True Diary of a Part-Time Indian
- 4 students had read The Absolutely True Diary of a Part-Time Indian and The Hate U Give
- 1 student had read The Hate U Give and When Everything Feels Like the Movies
- 1 student had read all three books

a) Complete a Venn diagram to represent this scenario.

b) If one student was chosen at random, what is the probability that the student did not read any of the books?

Probability Unit – Contingency Tables

5.5 CONTINGENCY TABLES

Contingency tables are useful when pieces of information fall in two categories and each category has subcategories. Let's say question 1 had two options: A and B, and question 2 had 2 options, X and Y. A contingency table allows us to present the results of both questions, as well as how the two questions relate to each other. That is, how many people chose $A \cap X, A \cap Y, B \cap X, B \cap Y$.

Ex: Dr. James asked her class of 25 two questions: 1) Which do you prefer: blue, red, yellow? 2) Do you like summer or winter better? The results are presented in the contingency table below

Season ↓ Colour →	BLUE	YELLOW	RED	TOTAL
SUMMER	7	2	6	15
WINTER	3	6	1	10
TOTAL	10	8	7	25

From a contingency table, we can determine probabilities.

$P(B \cap S) = \frac{7}{25}$ $\quad P(R \cap W) = \frac{1}{25}$ $\quad P(S) = \frac{15}{25} = \frac{3}{5}$ $\quad P(Y) = \frac{8}{25}$

Sometimes we will have to complete a contingency table before using it to determine probabilities.

Ex: A school is hosting a trip for grade 10 and grade 11 students. The possible destinations are: Banff, Vancouver, and Halifax. The teachers collected the following information.

	Banff	Vancouver	Halifax	TOTAL
Grade 10	90	225 − 125 100	250 − 90 − 100 60	250
Grade 11	300 − 90 210	125	500 − 210 − 125 165	750 − 250 500
TOTAL	300	225	750 − 300 − 225 225	750

Additionally, the teachers know the following: 250 total in Gr 10 $\quad 250 \times \frac{9}{25} = 90$

- Given that a student is in Grade 10, the probability they choose Banff is $\frac{9}{25}$
- The probability that the student is in Grade 11 given that they chose Vancouver is $\frac{5}{9}$ $\quad 225 \times \frac{5}{9} = 125$

225

a) What is the probability that a student, chosen at random, is in grade 10 and choose Halifax? $P(H \cap 10) = \frac{60}{750} = 2/25$

b) Given that a student chose the Banff trip, what is the probability that they are in Grade 10?

SKIP

Probability Unit – Contingency Tables
Practice Questions

1) A group of 115 students were surveyed about what board games they would like to play during a game night, and the snacks they would like to eat. The results are presented in the contingency table below.

	Pizza Rolls	Chips and Dip	Cookies	TOTAL
Scrabble	10	3	12	25
Trivial Pursuit	8	14	7	29
Monopoly	14	17	7	38
Risk	12	7	4	23
TOTAL	44	41	30	115

a) If a student was chosen at random, what is the probability they want to play Scrabble and want to eat Cookies?

b) If a student was chosen at random, what is the probability they want to play Risk?

c) If a student was chosen at random, what is the probability they want to eat Pizza Rolls or Chips and Dip

2) A school is planning the schedule for the following year. Students in Grade 11 must choose two options. They must choose an art (Art, Music, or Drama) and another option (Home Economics, Biology, or Law). The students returned their options sheets and we know the following:

- 50 students chose Music
- There are a total of 200 students entering grade 11 next year
- The probability that a student chose Art is $\frac{3}{5}$
- The probability that a student chose Home Economics is 0.3
- Given that a student chose Home Economics, the probability that the student chose Art is $\frac{1}{4}$
- The probability that a student chose Law, given that the student chose Art, is $\frac{2}{3}$
- The probability that a student chose Music, given that the student chose Home Economics is $\frac{5}{12}$
- No students chose both Drama and Law
- The probability a student chose Music and Law is 0.05

a) Complete a contingency table for the above scenario.

b) What is the probability that a student chose Biology?

c) What is the probability that a student chose both Drama and Home Economics?

Probability Unit – Conditional Probability

5.6 CONDITIONAL PROBABILITY

Conditional probability is the probability that an event will occur, given that another event has already occurred. We can determine the conditional probability using a contingency table or a formula.

To use a contingency table, focus attention on the row or column in the "given that" statement. Then only use those numbers to determine the possible ways the event can occur and the total number of possibilities.

Ex: Dr. James asked her class of 25 two questions: 1) Which do you prefer: blue, red, yellow? 2) Do you like summer or winter better? The results are presented in the contingency table below

Season ↓ Colour →	BLUE	YELLOW	RED	TOTAL
SUMMER	7	2	6	15
WINTER	3	6	1	10
TOTAL	10	8	7	25

a) If a student is chosen at random, what is the probability that the student prefers yellow given that the student likes winter best?

$P(Y|W) = \frac{6}{10}$

b) Given that a student likes Red best, what is the probability they also prefer summer?

$P(S|R) = \frac{6}{7}$

If we do not have a contingency table (and do not have enough information to construct one), we can use the following formula to determine conditional probability.

$$P(B\ given\ A) = P(B|A) = \frac{P(A \cap B)}{P(A)} \quad where\ P(A) \neq 0$$

$$\text{Or } P(B|A) \times P(A) = P(A \cap B)$$

Ex: Dr. James gave her class two tests. 62% (0.62) of the class passed both tests and 80% (0.80) of the class passed the first test. What percent of those who passed the first test also passed the second test?

$P(P_2|P_1) = \frac{P(P_1 \cap P_2)}{P(P_1)} = \frac{0.62}{0.80} = \frac{31}{40}$ or 0.775

$\times 100$

$= 77.5\%$

Probability Unit – Conditional Probability

Practice Questions

1) Canadian tourists coming back from a South American trip were interviewed about the countries they visited. Among the 40 tourists in the group:

- 20 visited Argentina
- 30 visited Brazil
- 12 visited both Argentina and Brazil

a) Given that a tourist visited Argentina, what is the probability they also visited Brazil?

b) Given that a tourist visited Brazil, what is the probability they also visited Argentina?

2) A group of 115 students were surveyed about what board games they would like to play during a game night, and the snacks they would like to eat. The results are presented in the contingency table below.

	Pizza Rolls	Chips and Dip	Cookies	TOTAL
Scrabble	10	3	12	25
Trivial Pursuit	8	14	7	29
Monopoly	14	17	7	38
Risk	12	7	4	23
TOTAL	44	41	30	115

Calculate:

a) $P(Scrabble|Cookies)$

b) $P(Pizza\ Rolls|Risk)$

d) $P(Monopoly|Chips\ and\ Dip)$

e) $P(Cookies|Trivial\ Pursuit)$

5.7 RANDOM EXPERIMENT WITH SEVERAL STEPS

Tree diagrams can help us determine the probability of an event when that event has multiple steps.

To create a tree diagram, we map out the possible options for the first step, then the second, etc. until all possible options are included. For example, let's create a tree diagram to map out the options of flipping a coin 3 times:

Step 1: List options (in this case, heads or tails) and associated probabilities

Step 2: Following each of the results in the first step, list all possible options for the second step (in this case, heads or tails)

Step 3: Following each of the results in the second step, list all possible options for the third step (in this case, heads or tails). Continue adding steps and branches to the tree as needed.

The probability of any given outcome is the product of each result needed to obtain that outcome.

$$P(ABC) = P(A) \times P(B) \times P(C)$$

Keep in mind that in this case, the probability of heads is the same as the probability for tails, but that need not need be the case.

FLIP 1	**FLIP 2**	**FLIP 3**	**OUTCOME**
HEADS $(P = 1/2)$	HEADS $(P = 1/2)$	HEADS $(P = 1/2)$	HHH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS $(P = 1/2)$	HHT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
	TAILS $(P = 1/2)$	HEADS $(P = 1/2)$	HTH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS $(P = 1/2)$	HTT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
TAILS $(P = 1/2)$	HEADS $(P = 1/2)$	HEADS $(P = 1/2)$	THH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS $(P = 1/2)$	THT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
	TAILS$(P = 1/2)$	HEADS $(P = 1/2)$	TTH $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$
		TAILS $(P = 1/2)$	TTT $\left(P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$

Probability Unit – Random Experiment With Several Steps

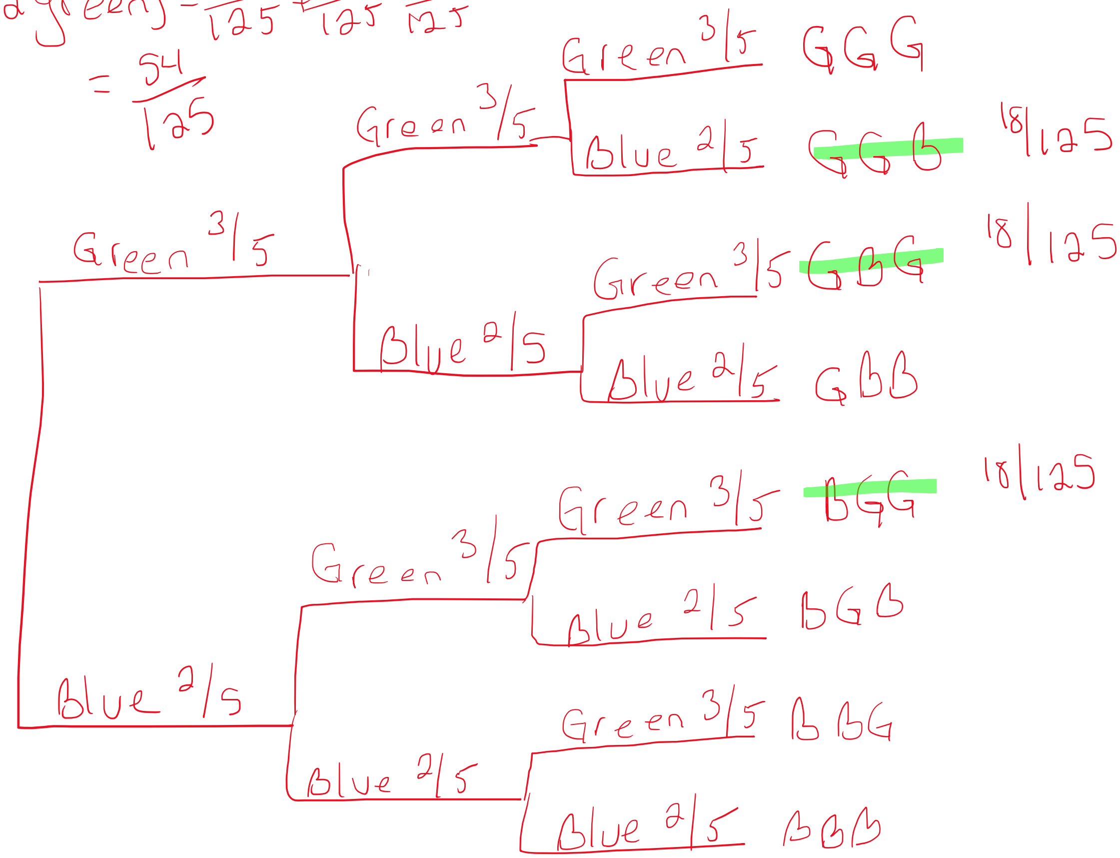
Ex: You are drawing marbles from a bag that contains 3 green marbles and 2 blue marbles. After you draw each marble you put it back in the bag. You draw 3 marbles.

a) What is the probability of drawing a green marble on the first draw, followed by a blue marble, followed by a green marble (that is, what is the probability of GBG)? $P(GBG) = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{18}{125}$

b) What is the probability of drawing exactly 2 green marbles?

$P(2\,green) = \frac{18}{125} + \frac{18}{125} + \frac{18}{125}$

$= \frac{54}{125}$



Probability Unit – Random Experiment With Several Steps

Ex: You are drawing marbles from a bag that contains 3 green marbles and 2 blue marbles. After you draw each marble you do not put it back in the bag. You draw 3 marbles.

a) What is the probability of drawing a green marble on the first draw, followed by a blue marble, followed by a green marble (that is, what is the probability of GBG)? $P(GBG) = 1/5$

b) What is the probability of drawing exactly 2 green marbles? $P(2\text{ green}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$

$P(GGB) = 3/5 \times \frac{2}{4} \times \frac{2}{3} = 1/5$

$P(GBG) = \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{12}{60} = \frac{1}{5}$

$P(BGG) = 2/5 \times 3/4 \times 2/3 = 1/5$

Probability Unit – Random Experiment With Several Steps

Practice Questions

1) A jar contains 5 red marbles, 3 yellow marbles, and 2 black marbles. Three marbles are drawn consecutively. Calculate the following probabilities:

a) one marble of each color is drawn if the marbles are drawn with replacement.

b) two red marbles and one black marble are drawn without replacement.

c) three marbles of the same color are drawn without replacement.

d) two black marbles and one yellow marble are drawn with replacement.

Probability Unit – Permutations and Combinations

5.8 PERMUTATIONS AND COMBINATIONS

Often, when we are talking outside of math class, we use the word combination in two different ways.

- We might say: **"My fruit salad has a combination of apples, grapes, and bananas."** But we could just as easily say: "My fruit salad has a combination of bananas, apples, and grapes" and we would be talking about the same fruit salad. When we list the fruit, **order does not matter**.

- We might say: **"The combination to the safe is 472."** If we said: "The combination to the safe is 247" that would be different. When we give this list of numbers, **order does matter**.

We will distinguish between these two possibilities:

- A **combination** is when **order doesn't matter**.
- A **permutation** is when **order does matter**.

Additionally, for each of these cases, sometimes a repetition will be allowed and sometimes a repetition will not be allowed. For example, maybe the lock is designed so that once a number is used it cannot be used again.

Below we will discuss each of the 4 cases:

	Order?	**Replacement?**	**Formula** ***n*** is the number of things to choose from and we are choosing ***r*** times	**Calculator**
Permutation	It matters	Allowed	n^r	Use exponent button
	It matters	Not Allowed	$\frac{n!}{(n-r)!}$	nPr button
Combination	Doesn't matter	Not Allowed	$\frac{n!}{r!\,(n-r)!}$	nCr button
	Doesn't matter	Allowed	$\frac{(r+n-1)!}{r!\,(n-1)!}$	Use algebra and the ! button on your calculator

Permutations

A **permutation** is an **ordered** combination. There are two types of permutations

- **Replacement is allowed** – In the safe example above, the "combination" to unlock the safe could be 444 – even though the 4 was used as the first number, it can be re-used.

- **Replacement is not allowed** – If there are 3 people running a race and we are interested in the possible ways they might place. If a person came in first, that person could not also come in second or third.

Probability Unit – Permutations and Combinations

Permutations with Replacement

When repetition is allowed (with replacement) and order matters, the formula is:

$$number\ of\ permutations\ (with\ replacement) = n^r$$

where ***n*** is the number of things to choose from and we are choosing ***r*** times.

Ex: In the lock example above, there are 10 digits to choose from {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and we choose 3 of them.

Calculate the number of possible "combinations" to the lock.

# of permutations $= 10^3 = 1000$ combinations to the lock

Permutations without Replacement

In this case, there are fewer possibilities, because we cannot re-use the number once it's been used.

In general, when order matters and repetition is not allowed (without replacement), we can determine the number of possibilities using the following:

$$(\#\ of\ elements\ for\ 1st\ position) \times (\#\ of\ elements\ for\ 2nd\ position) \times \ldots \times (number\ of\ elements\ for\ last\ position)$$

We can use something called the factorial function to write this more easily. The factorial function (symbol: !) just means to multiply a series of descending natural numbers. (The ! button is also on most calculators).

Ex: $4! = 4 \times 3 \times 2 \times 1 = 24$

We can write the formula for determining the number of possibilities when order matters and repetition is not allowed (without replacement) using the formula:

$$\frac{n!}{(n-r)!}$$

Where n is the number of things to choose from and we are choosing r times.

Note: You can also use the nPr button on your calculator.

Ex: A group of 10 people are running a race. How many possible ways could first, second, and third place be awarded?

# of permutations $= \frac{10!}{(10-3)!} = \frac{10!}{7!} = 720$ ways 1^{st}, 2^{nd}, & 3^{rd} could be awarded

Probability Unit – Permutations and Combinations

Combinations without Replacement

For combinations without repetition, think about a lottery where numbers are drawn one at a time and if we have all the numbers that are drawn (regardless of the order), we win!

We can write the formula for determining the number of possibilities when order does not matter and repetition is not allowed (without replacement) using the formula:

$$\frac{n!}{r!\,(n-r)!}$$

Where *n* is the number of things to choose from and we are choosing *r* times.

Note: You can also use the nCr button on your calculator.

Ex: There are 30 students in a classroom and the teacher wants to create one group of 3 students to decorate the bulletin board. How many different possible groups are there?

$\#$ of combinations $= \frac{30!}{3!(30-3)!} = \frac{30!}{3!\,27!} = 4060$ possible groups

Combinations with Replacement

For combinations with replacement, think about going for ice cream. The ice cream shop offers 10 different flavors of ice cream and you are going to get 3 scoops in a dish. In this case, maybe you want all one flavor. Maybe you want 2 scoops of 1 flavor and 2 of another. Or maybe you want 3 scoops of different flavors. In this case it won't matter which scoop is dished out first, second, or third, and there is replacement, because the same flavor can be used more than once.

We can write the formula for determining the number of possibilities when order does not matter and repetition is not allowed (without replacement) using the formula:

$$\frac{(r+n-1)!}{r!\,(n-r)!}$$

Where *n* is the number of things to choose from and we are choosing *r* times.

Ex: The ice cream shop offers 10 different flavors of ice cream and you are going to get 3 scoops in a dish. How many different combinations of ice cream could you create? If 3 scoops were randomly selected, what is the probability that all 3 scoops are the same flavor?

Combinations $= \frac{(3+10-1)!}{3!\,(10-3)!} = \frac{12!}{3!\,7!} = 15840$ possible combinations

10 favors, so 10 of those combinations have all 3 scoops same flavor

$P(\text{same flavor}) = \frac{10}{15840} = \frac{1}{1584}$

Probability Unit – Permutations and Combinations

Practice Questions

1) For an electronics project, Keyla must place 6 LEDs next to one another in a line. Keyla has packages of red, green, yellow, and blue LEDs. How many different possible arrangements could Keyla make?

2) You have placed 15 different songs into your favorites list on your phone. Your phone will randomly generate playlists of 6 songs where no song is repeated within each playlist. How many different playlists can your phone generate?

3) Dr. James is putting together treats for her class. She has purchased bags of five different flavored candies: cherry, grape, orange, green apple, and pink lemonade. Dr. James randomly gives 3 candies to each student.

a) How many different arrangements of candies are possible?

b) What is the probability that a student would receive 3 grape candies?

c) What is the probability that a student would receive 3 candies of the same flavor?

4) 10 marbles are placed in a bag. The marbles are numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. A student is asked to randomly pick 2 marbles simultaneously.

a) How many different arrangements of numbers could the student pick?

b) What is the probability that a student would pick two consecutive numbers (ex: 1 and 2 or 6 and 7)?

Probability Unit – Odds

5.9 ODDS

Odds and probability are related concepts. We must between **odds for** and **odds against**.

The **odds for** a particular event to occur is the ratio of the number of favorable outcomes to the number of unfavorable outcomes.

Odds for Event A
$Odds\ for\ Event\ A = \dfrac{The\ number\ of\ ways\ event\ A\ occurs}{The\ number\ of\ ways\ A\ does\ not\ occur}$ OR $= \dfrac{Number\ of\ favorable\ outcomes}{Number\ of\ unfavorable\ outcomes}$ OR $Number\ of\ favorable\ outcomes: Number\ of\ unfavorable\ outcomes$

Ex: The odds that a football team will win the championship are 3:2. This means that the team has 3 chances of winning and 2 chances of losing.

The **odds against** a particular event to occur is the ratio of the number of unfavorable outcomes to the number of favorable outcomes.

Odds against Event A
$Odds\ against\ Event\ A = \dfrac{The\ number\ of\ ways\ event\ A\ does\ not\ occur}{The\ number\ of\ ways\ A\ occurs}$ OR $= \dfrac{Number\ of\ unfavorable\ outcomes}{Number\ of\ favorable\ outcomes}$ OR $Number\ of\ unfavorable\ outcomes: Number\ of\ favorable\ outcomes$

Ex: The odds against a player who bets on "0" in roulette are 36:1. That means the player has 36 chances of losing and 1 chance of winning.

Probability Unit – Odds

Difference between probability and odds

Probability of Event A: P(A)
$P(A) = \frac{\text{The number of ways event } A \text{ can occur}}{\text{The total number of possible outcomes}}$ $= \frac{\text{number of favorable outcomes}}{\text{number of favorable outcomes} + \text{number of unfavorable outcome}}$

Ex: The odds that the basketball team will win the next game are 4 to 3. What is the probability that the basketball team will NOT win the next game?

odds for $= 4:3$ $\quad P(\text{win}) = \frac{4}{4+3} = \frac{4}{7}$

so $P(\text{loss}) = 1 - \frac{4}{7} = \frac{3}{7}$

Practice Questions

1) The probability that the local basketball team will win the next game is estimated at $\frac{2}{7}$.

a) What are the odds for the team winning the next game?

b) What are the odds against the team winning the next game?

2) Two dice (numbered 1 to 6) are rolled. What are:

a) the odds for obtaining a sum of 7?

b) the odds against obtaining a sum of 12?

c) the odds for obtaining a sum of 2?

d) the odds for obtaining a sum of 1?

Probability Unit – Expectation and Fairness

5.10 EXPECTATION AND FAIRNESS

We can use mathematical **expectation** to determine how much money a person would expect to win (or lose) when placing a bet, given they know the probability of winning (and of losing). We also use the term "**expected gain**" interchangeably with expectation.

- If the result is positive, the person can expect to make a profit.
- If the result is negative, the person can expect to lose money.

$$Expected\ gain = (probability\ of\ winning) \times \underbrace{(net\ gain)}_{Gain\ -\ initial\ bet} + (probability\ of\ losing) \times \underbrace{(loss)}_{initial\ bet\ (written\ as\ negative)}$$

Ex: In a game of roulette, a person who has bet on the winning number receives 35 times the amount of the bet. Since the roulette wheel slots are numbered 0 to 36, the probability that the ball will land on a given number is $\frac{1}{37}$.

Determine the expected gain for a bet of \$10.

$$\text{Expected Gain} = \left(\frac{1}{37}\right)(350-10) + \left(\frac{36}{37}\right)(-10)$$

$$= \left(\frac{1}{37}\right)(340) + \left(\frac{36}{37}\right)(-10)$$

$$= \frac{340}{37} - \frac{360}{37}$$

$$= -\frac{20}{37} = -\$0.54$$

Probability Unit – Expectation and Fairness

Fairness

- When the mathematical expectation (or expected gain) of a game is greater than 0, it is advantageous to the person playing.
- When the mathematical expectation (or expected gain) of a game is less than 0, it is disadvantageous to the person playing.
- When a game's mathematical expectation is equal to 0, the game is considered **fair**. That is, given a person plays an infinite number of times, they can expect to lose nothing and gain nothing.

Ex: A game consists of drawing a marble at random from a set composed of 5 red marbles and 4 blue marbles. It costs a person \$10 to place a bet. If a blue marble is drawn, the person wins \$12.50 plus their initial bet. If a red marble is drawn, the \$10 bet is lost. Is this game fair?

$$\text{Expected Gain} = \left(\frac{4}{9}\right)(22.50-10) + \left(\frac{5}{9}\right)(-10)$$

$$= \left(\frac{4}{9}\right)(12.50) + \left(\frac{5}{9}\right)(-10)$$

$$= \frac{50}{9} + \frac{-50}{9}$$

$$= 0$$

Yes, the game is fair

Ex: For \$2, a person can participate in a draw for a \$50 gift card. The probability of winning the draw is $\frac{1}{100}$. Is this draw fair?

$$\text{Expected Gain} = \left(\frac{1}{100}\right)(50-2) + \left(\frac{99}{100}\right)(-2)$$

$$= \left(\frac{1}{100}\right)(48) + \left(\frac{99}{100}\right)(-2)$$

$$= \frac{48}{100} + \frac{-198}{100}$$

$$= \frac{-150}{100} = -\$1.5$$

No, the game is <u>not</u> fair

Probability Unit – Expectation and Fairness

Practice Questions

1) A carnival game costs $2 to play. In this game, there is a jar containing 2 black marbles and 3 white marbles. If you draw a black marble, you get $4. If you draw a white marble you get nothing. What is the expected gain from this game?

2) Consider the following game. Toss a coin twice. You lose your initial bet of $10 if you do not get tails on either toss. You win $4 and get your initial bet back if you get tails on one toss. You win $8 and get your initial bet back if you get tails on both tosses. What is the expected gain from this game?

3) In a game at a carnival, a person must draw a card from a deck of 54 cards (the deck contains both jokers). If the person draws one of the four aces, they win a box of candy valued at $10. If it costs $2 to play, is this game fair?

4) In a game at a carnival, a person must draw a card from a deck of 54 cards. If the person draws one of the four aces, they win a box of candy valued at $10. If the person draws one two jokers, they win a teddy bear valued at $40. If the person draws any other card, they do not win anything.

a) If it costs $2 to play, is this game fair?

b) If the game is not fair, how much should the game cost to play in order for it to be fair?

Probability Unit – Weighted Mean
5.11 WEIGHTED MEAN

Remember that the mean of a set of numbers is the average. To determine mean:

$$Mean = \frac{sum\ of\ data\ values}{total\ number\ of\ values}$$

Add all the numbers together and divide by how many numbers there are.

Ex: A student received the following grades on their math quizzes: 67%, 72%, 85%, 49% and 65%. What is the student's average quiz grade in math?

$$\text{Avg. Grate} = \frac{67+72+85+49+65}{5} = \frac{338}{5} = 67.6\%$$

For a weighted mean, some values are more important than others.

To calculate weighted mean:

- Convert the weight of each item from a percent to a decimal (divide by 100)
- Multiply the value of each item by the weight
- Add (values x weight) together

Ex: A student has the following marks in math.

Item	Grade (%)	Weight (%)	
Test 1	72%	50	0.5
Quiz 1	64%	25	0.25
Homework 1	81%	25	0.25

What is the student's overall math mark?

$$\text{Mark} = (72)(0.5) + (64)(0.25) + (81)(0.25)$$
$$= 36 + 16 + 20.25$$
$$= \underline{72.25\%}$$

Probability Unit – Weighted Mean

Practice Questions

1) In order to calculate final grades in a class, the teacher combines assignment grades, test grades, and quiz grades. Assignments are worth 30% of the final grade, tests are worth 50%, and quizzes are worth 20%. If a student has an assignment grade of 82%, a test grade of 65%, and a quiz grade of 74%, what is the student's final grade?

2) Nico wants to buy a new car and has decided to rate each car according to the following criteria and values:

- Appearance: 15%
- Reliability: 40%
- Efficiency: 15%
- Comfort: 30%

The following is a list of possible cars and the ratings of each (0 is the lowest mark and 10 is the highest).

	Appearance	Reliability	Efficiency	Comfort
Car A	5	9	8	6
Car B	6	10	8	4
Car C	4	7	3	9

Nico has decided to buy the car with the highest average rating. Which car should Nico buy?

5.12 PROBABILITY EXAM STYLE QUESTIONS

Multiple Choice

1) Before going into their last game of the season, the high school rugby team had won five games and lost three games.

Given their record, what are the odds in favor of winning their last game?

A) 5 to 3

B) 3 to 5

C) 5 to 8

D) 3 to 8

2) A gumball machine contains 120 gumballs that are either blue or red.

If a gumball is drawn at random from the machine, the odds that the gumball will be blue are 3 to 5.

How many blue gumballs are there in the gumball machine?

A) 45

B) 48

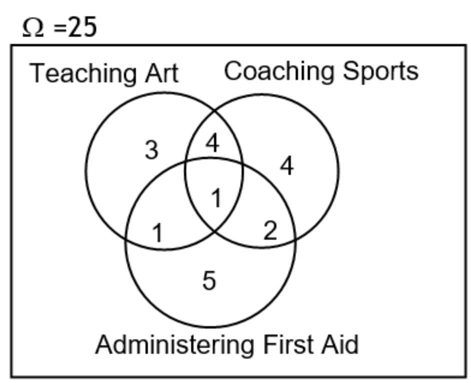
C) 72

D) 75

3) Counsellors are hired for a large summer camp; 25 students applied for the positions.

To be hired, they must be qualified in one, two or three areas of expertise:

- Teaching art
- Coaching sports
- Administering first aid



The diagram shows the number of students with each expertise.

Which of the following statements is true?

A) The number of students that have expertise in teaching art and coaching sports is 1.

B) The number of students that have expertise in coaching sports or administering first aid is 17.

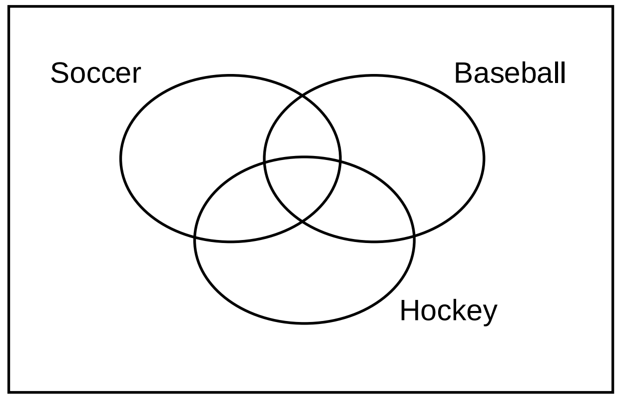
C) The number of students that have expertise in teaching art, administering first aid and coaching sports is 8.

D) The number of students that have no expertise in these areas is 4.

Short Answer

4) A school offers three sports for the Sec V students to play: soccer, baseball and hockey. There are 100 Sec V students. A student can choose to play more than one sport. Here is a list of how many of those students signed up for the sports options:

- Soccer 42
- Baseball 46
- Hockey 45
- All 3 sports 20
- Soccer and baseball 30
- Soccer and hockey 24
- Baseball and hockey 29



a) How many students signed up only for hockey?

b) How many students did not sign up for any sports?

5) A survey was conducted at a nearby university to learn about the commuting habits of both students and employees. The respondents were asked whether they use public transit or drive a car. Given the respondent is a student, there is a probability of $\frac{4}{7}$ that he or she uses public transport.

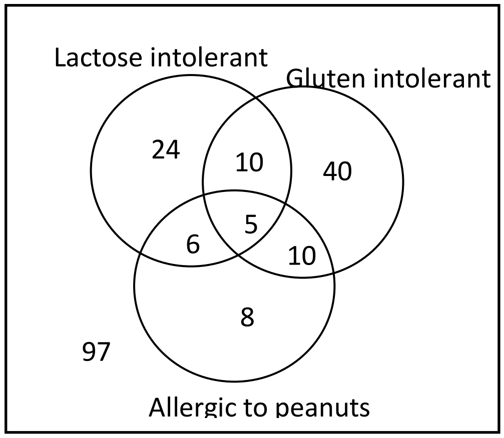
The incomplete table below illustrates additional findings of the survey.

	PUBLIC TRANSIT	CAR	TOTAL
STUDENT			3934
EMPLOYEE			1686
TOTAL	2856	2764	5620

a) How many students use public transit?

b) What is the probability that the respondent is a student, given that the respondent drives a car?

6) The school nurse asked 200 students if they had any dietary restrictions.
Their responses were recorded in the Venn diagram below.



One of the 200 students was chosen at random.

Given that the chosen student is gluten intolerant, what is the probability that this student is also lactose intolerant?

Long Answer

7) While at a fundraising event, Brian wants to pick a game that will allow him to win a prize. He has done some research and has decided to play either Die Roll or Roulette Wheel.

The amount of money Brian must bet to play either game is the same.

Die Roll

The player rolls an 8 sided die after placing his or her bet.

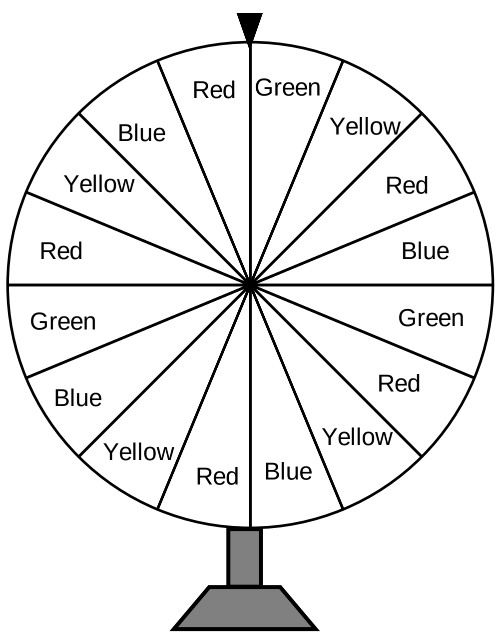
- If an odd number is rolled, the player wins nothing.
- If a 2, 4 or 6 is rolled, the player wins the value of the bet.
- If an 8 is rolled, the player wins $30.

This game is fair.

Roulette Wheel

The player places a bet and then spins the wheel. There are 16 congruent sections on the wheel.

- If the wheel stops on one of the 4 yellow sections, the player wins $6.
- If the wheel stops on one of the 5 red sections, the player wins nothing.
- If the wheel stops on one of the 4 blue sections, the player wins $8.
- If the wheel stops on one of the 3 green sections, the player wins $10.



Which game is more advantageous for Brian to play?

8) A school is offering 3 different trip destinations: Paris, Los Angeles and New York City. Students may only choose only 1 destination. The teachers have collected the following data based on destination and grade.

- There are 750 students who signed up.
- Twice as many grade 10 students as grade 11 students signed up.
- The probability that a student signed up for Paris is $\frac{2}{5}$.
- The probability that a student signed up for Los Angeles is the same as the probability that they signed up for New York City.
- Given that a student chose Paris, the probability that the student is in Grade 11 is 30%.
- Given that a student is in Grade 10, the probability that they signed up for Los Angeles is 0.25.

Given that a student chosen at random is in Grade 11, what is the probability that they signed up for New York City?

9) While at a carnival, Serena finds two games of chance that will allow her to win a prize. She decided to play either The Colour Is Right or The Fruit Wheel.

The amount of money Serena must bet to play either game is the same.

THE COLOUR IS RIGHT

After placing a bet, the player randomly selects a marble from a bag.

The bag contains 5 red marbles, 2 blue marbles and 3 yellow marbles.

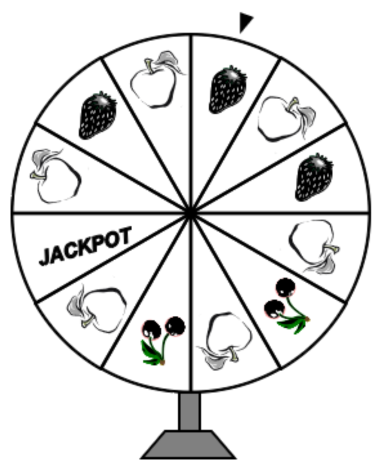
- If a red marble is chosen, the player does not receive any money.
- If a blue marble is chosen, the player receives $28.
- If a yellow marble is chosen, the player receives the value of the bet.

This game is fair.

THE FRUIT WHEEL

This game consists of spinning a wheel after placing a bet. The wheel is divided into 12 congruent sectors.

- If the pointer stops on an apple, the player receives $4.
- If the pointer stops on a strawberry, the player does not receive any money.
- If the pointer stops on a cherry, the player receives $12.
- If the pointer stops the JACKPOT section, the player receives $40.



Is The Fruit Wheel game to Serena's advantage, to Serena's disadvantage, or is it fair? Justify your reasoning.