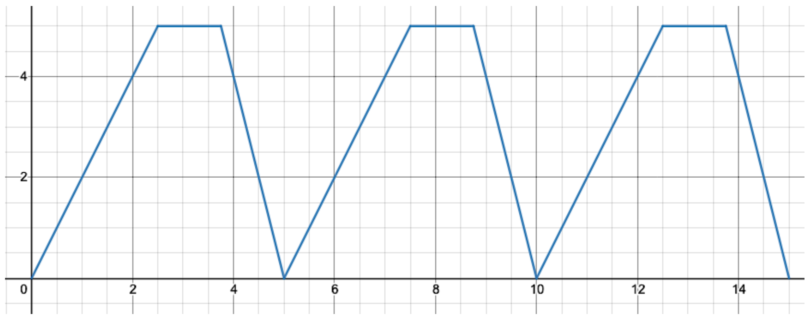
Trigonometric Functions

Sine Function

A **periodic function** repeats the same value at equal intervals of the independent variable (the pattern repeats).

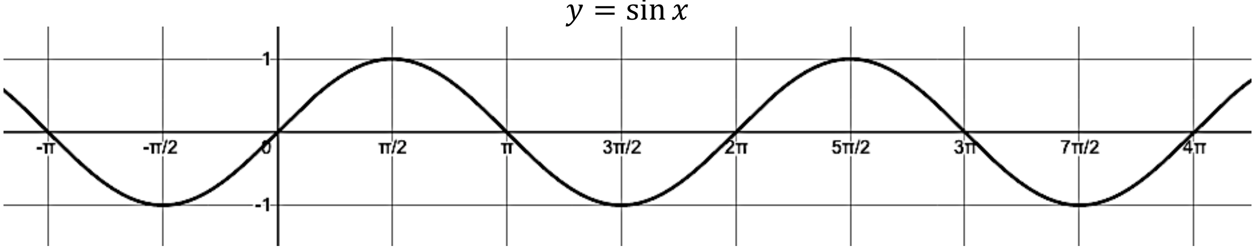
Ex:



A sine function is a periodic function. Knowing this will help graph and/or solve sine functions.

The basic rule of a sine function is:

$$y = \sin x$$



The **period** is the length of one cycle (2π)

The **frequency** is how often the cycle happens per unit time – the reciprocal of the period $\left(\frac{1}{2\pi}\right)$

The **amplitude** is the distance from the center to the highest (or lowest) point (1)

The **maximum** is 1. The **minimum** is -1. So the **range** is $[-1, 1]$

The **domain** is $]-\infty, \infty[$

The **zeros** are integer multiples of π

The transformed sine function is

$$y = a\sin(b(x-h)) + k$$

The **amplitude** is “a” and it is a vertical stretch and/or reflection over the x-axis.

$$a = \frac{fmax - fmin}{2}$$

The **frequency** is “b” and it is a horizontal stretch and/or reflection over the y-axis.

The **period** is calculated using $period = \frac{2\pi}{|b|}$

“h” is a horizontal translation (phase shift)

“k” is a vertical translation

(h,k) is a “starting” point on the curve

The **maximum** is $max = k + |a|$ and the **minimum** is $min = k - |a|$

If $ab > 0$, then start by going up from (h,k)
If $ab < 0$, then start by going down from (h,k)

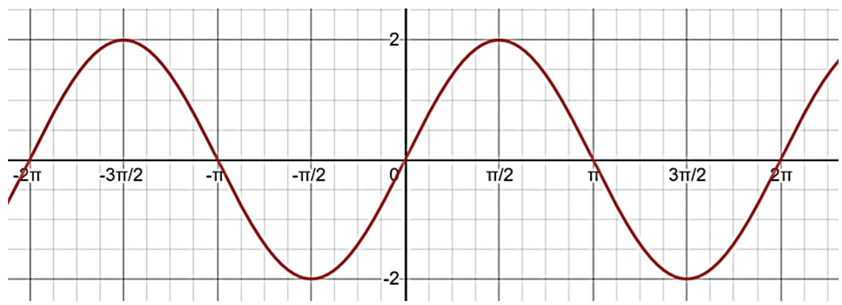
Sketching a Sine Function

To sketch a sine function:

1) Determine a, b, h, k
2) Find period, max, min, and starting point
3) Determine if going up or down from starting point
4) Use period and min/max to label graph
5) Plot starting point and additional points for one cycle
6) Sketch and graph additional cycles as necessary

Ex: Sketch the function $y = 2\sin x$

$a = 2, b = 1, h = 0, k = 0$



$min = k - |a|, max = k + |a|$

$min = 0 - |2|, max = 0 + |2|$

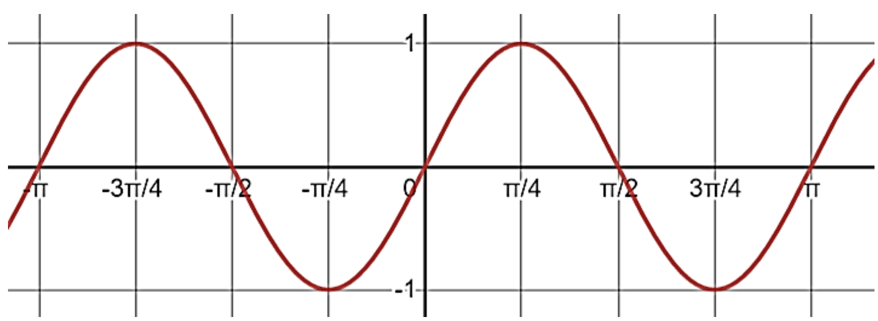
$min = -2, max = 2$

$$period = \frac{2\pi}{1} = 2\pi$$

$starting\ point: (0,0)\ ab > 0\ so\ increasing$

Ex: Sketch the function $y = \sin 2x$

$a = 1, b = 2, h = 0, k = 0$



$min = k - |a|, max = k + |a|$

$min = 0 - |1|, max = 0 + |1|$

$min = -1, max = 1$

$$period = \frac{2\pi}{2} = \pi$$

$starting\ point: (0,0)\ ab > 0\ so\ increasing$

Ex: Sketch the function $y = 3\sin(\frac{1}{2}(x - \pi)) + 1$

$a = 3, b = \dfrac{1}{2}, h = \pi, k = 1$

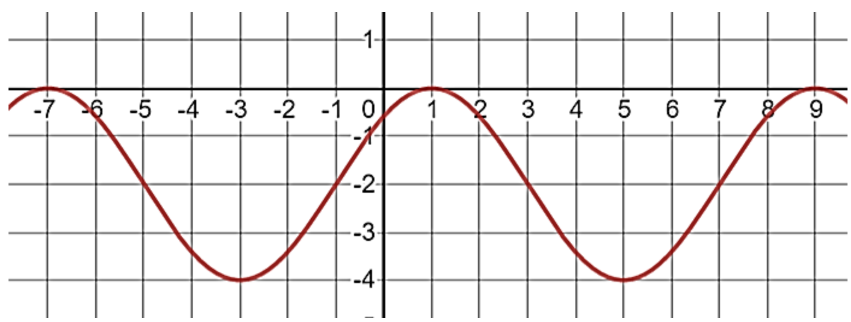
$min = k - |a|, max = k + |a|$
$min = 1 - |3|, max = 1 + |3|$
$min = -2, max = 4$

$period = \dfrac{2\pi}{1/2} = 4\pi$

$starting\ point: (\pi, 1\pi)\ ab > 0\ so\ increasing$

Ex: Sketch the function $y = -2\sin(\frac{\pi}{4}(x - 3)) - 2$

$a = -2, b = \dfrac{\pi}{4}, h = 3, k = -2$



$min = k - |a|, max = k + |a|$
$min = -2 - |-2|, max$
$= -2 + |-2|$
$min = -2 - 4, \quad max = 0$

$period = \dfrac{2\pi}{\pi/4} = 8$

$starting\ point: (3, -2)\ ab < 0\ so\ decreasing$

Solving Sine Functions

To solve for y given x, plug in x and solve (in radians)

Ex: Given $f(x) = 4\sin\left(\frac{\pi}{6}(x+1)\right) - 3$, determine the initial value of $f(x)$.

$$f(x) = 4\sin\left(\frac{\pi}{6}(x+1)\right) - 3$$
$$f(0) = 4\sin\left(\frac{\pi}{6}(0+1)\right) - 3$$
$$f(0) = 4\sin\left(\frac{\pi}{6}(1)\right) - 3$$
$$f(0) = 4\sin\left(\frac{\pi}{6}\right) - 3$$
$$f(0) = 4\left(\frac{1}{2}\right) - 3$$
$$f(0) = 2 - 3$$
$$f(0) = -1$$

To solve for x given y:

1. Isolate $\sin\big(b(x-h)\big)$ and let $\big(b(x-h)\big) = \theta$ if $b \neq 1$ *or* $h \neq 0$
2. Use CAST rule or unit circle to solve for θ
3. Let $\big(b(x-h)\big) = \theta$ and solve for x
4. Solve for x over the given domain

Ex: Solve $2\sin x - \sqrt{3} = 0$ over $0 \leq \theta \leq 2\pi$

$$2\sin x - \sqrt{3} = 0$$
$$2\sin x = \sqrt{3}$$
$$\sin x = \frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3}\ and\ \frac{2\pi}{3}$$

Ex: Solve $4\sin x + 3 = 0$ over $0 \leq \theta \leq 2\pi$

$$4\sin x + 3 = 0$$
$$4\sin x = -3$$

$\sin x = -\frac{3}{4}$ in quadrants 3 and 4

$$\theta_R = 0.8481$$
$$\theta_1 = \pi + 0.8481 = 3.9897$$
$$\theta_2 = 2\pi - 0.8481 = 5.4351$$

So $x = 3.9897\ and\ 5.4351$

Ex: Solve $\sin x + 6 = 2\ 0$ over $0 \le \theta \le 2\pi$

$$\sin x + 6 = 2$$
$$\sin x = -4$$

NO SOLUTION

Ex: Solve $3\sin(x-\pi) + 3 = 0$ over $0 \le \theta \le 8\pi$

$$3\sin(x-\pi) + 3 = 0$$
$$3\sin(x-\pi) = -3$$
$$\sin(x-\pi) = -1$$
$$\sin\theta = -1$$
$$\theta = \frac{3\pi}{2}$$
$$x - \pi = \frac{3\pi}{2}$$
$$x = \frac{5\pi}{2}$$

$$period = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$$

So

$$x = \frac{5\pi}{2}, \frac{5\pi}{2} + 2\pi, \frac{5\pi}{2} + 4\pi$$

Also

$$x = \frac{5\pi}{2} - 2\pi$$

Therefore:

$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$$

Ex: Solve $6\sin\big(\pi(x-1)\big) = -3$

$$6\sin\big(\pi(x-1)\big) = -3$$
$$\sin\big(\pi(x-1)\big) = -\frac{1}{2}$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6}\ and\ \frac{11\pi}{6}$$
$$\pi(x-1) = \frac{7\pi}{6}\ \ and\ \pi(x-1) = \frac{11\pi}{6}$$
$$x - 1 = \frac{7}{6}\ \ and\ x - 1 = \frac{11}{6}$$
$$x = \frac{13}{6}\ \ and\ x = \frac{17}{6}$$

$$period = \frac{2\pi}{|b|} = \frac{2\pi}{\pi} = 2$$

Therefore:

$$x = \frac{13}{6} + 2n\ and\ x = \frac{17}{6} + 2n$$

Ex: Given $f(x) = 4\sin\left(\frac{\pi}{6}(x+8)\right) + 4$ over a domain of $[0, 30]$

For what values of x is $y = 6$?

$$4\sin\left(\frac{\pi}{6}(x+8)\right) + 4 = 6$$

$$4\sin\left(\frac{\pi}{6}(x+8)\right) = 2$$

$$\sin\left(\frac{\pi}{6}(x+8)\right) = \frac{1}{2}$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6} \; and \; \frac{5\pi}{6}$$

$$\frac{\pi}{6}(x+8) = \frac{\pi}{6} \;\; and \;\; \frac{\pi}{6}(x+8) = \frac{5\pi}{6}$$

$$x + 8 = 1 \;\; and \; x + 8 = 5$$

$$x = -7 \;\; and \; x = -3$$

$$period = \frac{2\pi}{|b|} = \frac{2\pi}{\pi/6} = 12$$

$$x = -7 + 12 = 5$$
$$x = 5 + 12 = 17$$
$$x = 17 + 12 = 29$$

$$x = -3 + 12 = 9$$
$$x = 9 + 12 = 21$$

Therefore:

$$x = 5, 9, 17, 21, 29$$

Sine Function Inequalities

Solving sine function inequalities is very similar to solving sine functions. The only difference is we will need to look at a sketch (or use a test point) to determine the intervals for the solution.

Steps:

1) Change inequality to equality
2) Solve as before
3) Use sketch or test point to determine solution interval
4) Solve over domain (or generalize if no domain is given)

Ex: Where is the function $f(x) = -\sin 3x + \frac{1}{2}$ greater than 0?

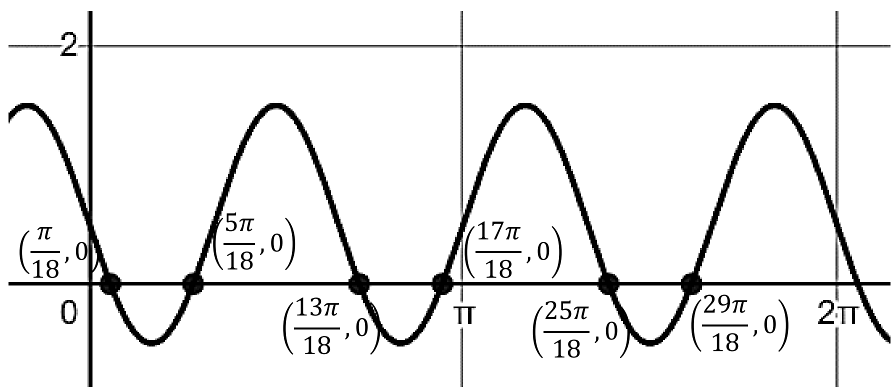
a) Over $0 \leq x \leq 2\pi$
b) Over $-\infty \leq x \leq \infty$

SOLVE

$$-\sin 3x + \frac{1}{2} > 0$$
$$-\sin 3x + \frac{1}{2} = 0$$
$$-\sin 3x = -\frac{1}{2}$$
$$\sin 3x = \frac{1}{2}$$

$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6} \; and \frac{5\pi}{6}$$
$$3x = \frac{\pi}{6} \;\; and \; 3x = \frac{5\pi}{6}$$
$$x = \frac{\pi}{18} \;\; and \; x = \frac{5\pi}{18}$$

SKETCH



Use period to solve for all critical points.

$$Period = \frac{2\pi}{3} = \frac{12\pi}{18}$$

Consult sketch to determine possible intervals.

SOLUTION

a) $\left[0, \frac{\pi}{18}\right[\cup \left]\frac{5\pi}{18}, \frac{13\pi}{18}\right[\cup \left]\frac{17\pi}{18}, \frac{25\pi}{18}\right[\cup \left]\frac{29\pi}{18}, 2\pi\right]$

b) $\left]\frac{5\pi}{18} + \frac{12\pi}{18}n, \frac{13\pi}{18} + \frac{12\pi}{18}n\right[$

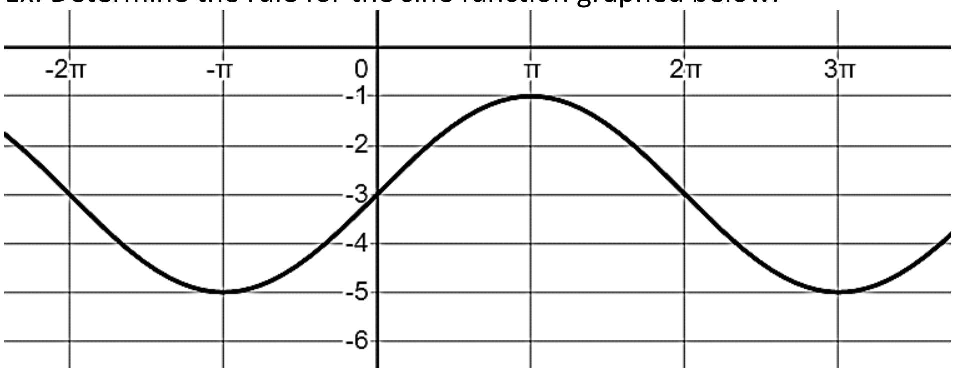
Finding the Rule of Sine Functions From a Graph

To find the rule of a sine function from a graph:

1) Find the center line (k): $\frac{\max+\min}{2}$
2) Find h (use any point where $y = k$)
3) Determine period (one full cycle) and b: $\frac{2\pi}{period}$
4) Determine a: $max - k$
5) Determine signs on a and b (does the function increase or decrease from h,k)
6) Write the function

Note: There are an infinite number of possibilities when writing the rule based on the h,k you choose and the signs on a and b.

Ex: Determine the rule for the sine function graphed below.



$$k = \frac{-1 - -5}{2} = -3$$

$$Period = 4\pi$$

$$b = \frac{2\pi}{4\pi} = \frac{1}{2}$$

Choose h at 2π

$$|a| = max - k = -1 - -3 = 2$$

$$(h, k) = (2\pi, -3)$$

Function decreases from (h,k) so a and b must have opposite signs

$$\therefore f(x) = 2\sin\left(-\frac{1}{2}(x - 2\pi)\right) - 3$$

Finding the Rule of Sine Functions From Words
Recall $y = a\sin[b(x-h)] + k$

Where:
a is the **amplitude** (a vertical stretch and/or reflection over the x-axis) $a = \frac{fmax - fmin}{2}$

b is the **frequency** (a horizontal stretch and/or reflection over the y-axis) $period = \frac{2\pi}{|b|}$

h is the horizontal translation/phase shift (the x-value when $y = k$)

"k" is the vertical translation $k = \frac{fmax + fmin}{2}$

(h,k) is a "starting" point on the curve

The **maximum** is $max = k + |a|$ and the **minimum** is $min = k - |a|$

If $ab > 0$, then start by going up from (h,k)
If $ab < 0$, then start by going down from (h,k)

Ex: Lobster fishermen dock at a small fishing harbor on the Bay of Fundy. The depth of the water varies according to the tides. A sinusoidal function can be used to predict the water's depth.

The Bay of Fundy has the highest tidal range in the world. At low tide, the depth of the water is 2m. Six hours later, at high tide, the depth of the water is 14m.

The fishermen leave from, and return to, the harbor when the depth of the water is at least 5 m.

For how many hours can the fishermen be at sea between two consecutive low tides?

$$a = \frac{fmax - fmin}{2} = \frac{14-2}{2} = 6$$

$$period = 12 \; so \; b = \frac{2\pi}{12} = \frac{\pi}{6}$$

h: 12 hour period. If low tide is at $x = 0$

$$h = \frac{12}{4} = 3$$

$$k = \frac{fmax + fmin}{2} = \frac{14+2}{2} = 8$$

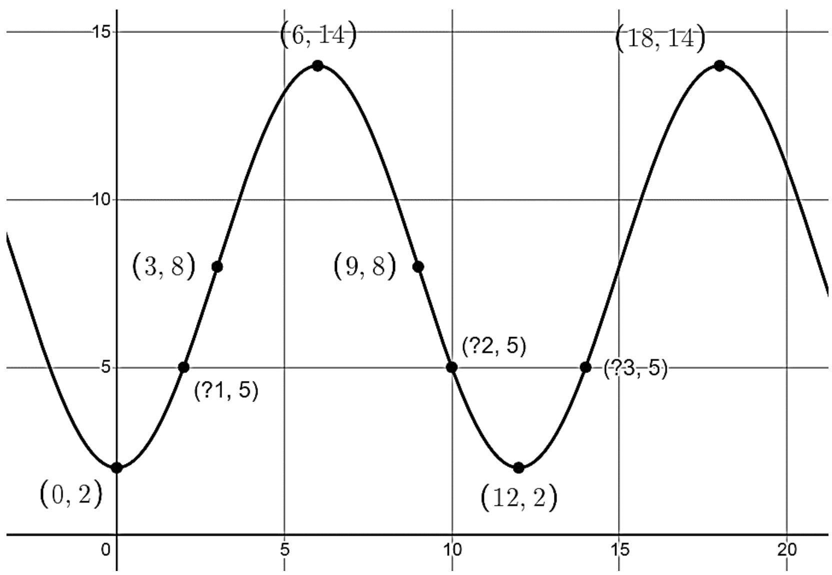
Increasing to start, so a and b same signs

$$f(x) = 6\sin\left[\frac{\pi}{6}(x-3)\right] + 8$$

Now we can use the function to determine when the water depth is at least 5 m.

$$f(x) = 6\sin\left[\frac{\pi}{6}(x-3)\right] + 8$$

$$6\sin\left[\frac{\pi}{6}(x-3)\right] + 8 \geq 5$$
$$6\sin\left[\frac{\pi}{6}(x-3)\right] + 8 = 5$$
$$6\sin\left[\frac{\pi}{6}(x-3)\right] = -3$$
$$\sin\left[\frac{\pi}{6}(x-3)\right] = -\frac{1}{2}$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6} \ and \frac{11\pi}{6}$$
$$\frac{\pi}{6}(x-3) = \frac{7\pi}{6} \quad and \ \frac{\pi}{6}(x-3) = \frac{11\pi}{6}$$
$$(x-3) = 7 \quad and \ (x-3) = 11$$
$$x = 11 \quad and \ x = 14$$



Looking at our solutions, we know $?2 = 11 \ and \ ?3 = 14$.
We also know $?1 = ?3 - one\ period = 2$

The question states that the fishermen leave and return between two consecutive low tides, so they are never at sea during a low tide.
means they must leave at ?1 (2) and return at ?2 (10).

The fishermen are away for 8 hours.

Cosine Functions

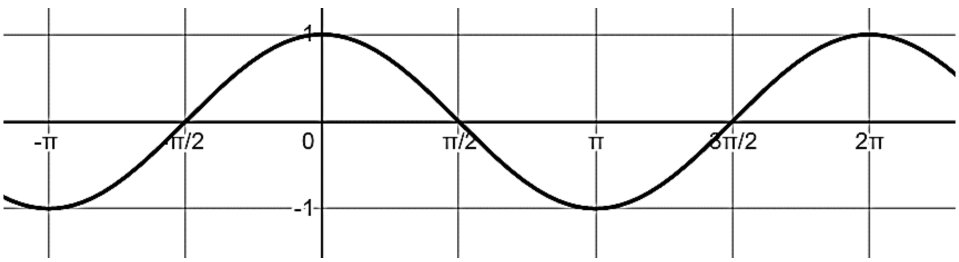
A cosine function is a sine function shifted by $\frac{\pi}{2}$

$$\cos x = \sin\left(x + \frac{\pi}{2}\right) \;\; or \cos x = \sin\left(x - \frac{3\pi}{2}\right)$$

$$\sin x = \cos\left(x - \frac{\pi}{2}\right) \;\; or \;\; \sin x = \cos\left(x + \frac{3\pi}{2}\right)$$

Basic function

$$f(x) = \cos x$$



Domain: $]-\infty, \infty[$

Range: $[-1, 1]$

Period: 2π

Amplitude: 1

Initial Value: 1

Parameters (a, b, h, k) are the same as in the sine function:
a: amplitude
b: $\frac{2\pi}{period}$
h: horizontal translation (phase shift)
k: vertical translation

What's new?

1) The sign of b has no effect on the graph since the graph is symmetrical over the y-axis.
    a. If $a > 0$, decreasing to "start" ("start" at max)
    b. If $a < 0$ increasing to "start" ("start" at min)
2) (h, k) is on the graph
    a. If "start" is max, then "starting point" is $(h, \max) = \mathrm{h}, \mathrm{k} + |a|)$
    b. If "start" is min, then "starting point" is $(h, \min) = \mathrm{h}, \mathrm{k} - |a|)$

Tangent Function

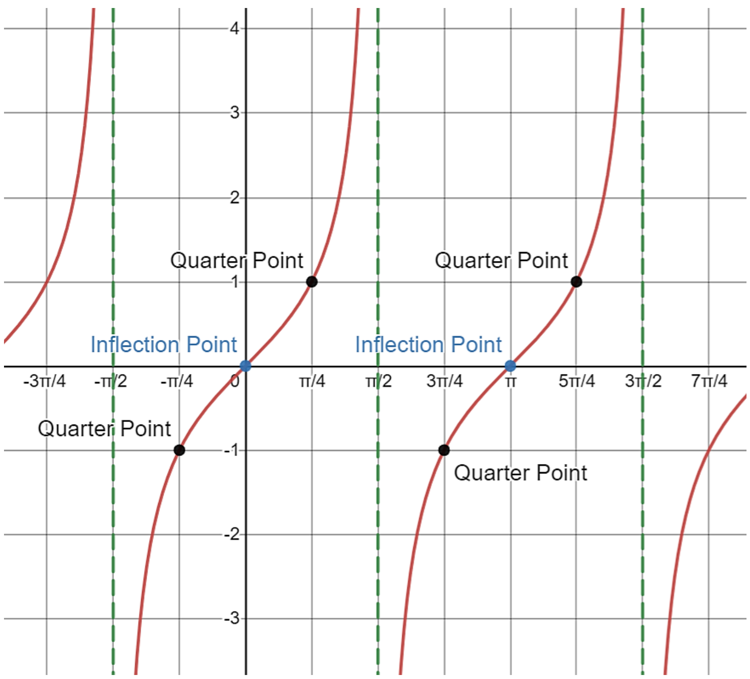
Recall that $\tan x = \frac{opp}{adj}$, but we also know that $\tan x = \frac{\sin x}{\cos x}$

Basic Function

If we look at the unit circle, we find that $\tan x = 1\ at\ \frac{\pi}{4}\ and\ \frac{5\pi}{4}$ and $\tan x = -1\ at\ \frac{3\pi}{4}\ and\ \frac{7\pi}{4}$

We also know that $\tan x$ is 0 when $\sin x = 0\ \ (0\ and\ \pi)$.

We can also determine that $\tan x$ is undefined when $\cos x = 0\ \left(\frac{\pi}{2}\,and\,\frac{3\pi}{2}\right)$. This means there must be vertical asymptotes at $\frac{\pi}{2}\,and\,\frac{3\pi}{2}$.



Key Information

$Period = \pi$

$Domain: \mathbb{R}\ /\left\{\frac{\pi}{2} + \pi n\right\}$

$Range: \mathbb{R}$

$Asymptotes: \frac{\pi}{2} + \pi n$

$Zeros: \pi + \pi n$

$Initial\ Value: 0$

No max, min, or amplitude

Transformed Tangent Function

$$f(x) = a\tan\big(b(x-h)\big) + k\quad or\ f(x) = a\left(\frac{\sin(b(x-h))}{\cos\big(b(x-h)\big)}\right) + k$$

If $ab > 0$ the function is increasing
If $ab < 0$ the function is decreasing

$Period = \frac{\pi}{|b|}$

$Asymptotes: solve\ \cos\big(b(x-h)\big) = 0$

Sketching a Tangent Function

To sketch a tangent function:

- Find inflection point (h, k)
- Find period: $\frac{\pi}{|b|}$
- Find asymptotes: $\cos\big(b(x-h)\big) = 0$
- Find zeros
- Determine increasing/decreasing
- Sketch

Ex: Sketch $f(x) = 2\tan\big(2(x-\pi)\big)$

Inflection point $(\pi, 0)$

Period: $\frac{\pi}{2}$

Asymptotes: $\cos\big(2(x-\pi)\big) = 0$

$2(x-\pi) = \frac{\pi}{2}$ and $2(x-\pi) = \frac{3\pi}{2}$

$(x-\pi) = \frac{\pi}{4}$ and $(x-\pi) = \frac{3\pi}{4}$

$x = \frac{5\pi}{4}$ and x= $\frac{7\pi}{4}$

So asymptotes at: $x = \frac{5\pi}{4} + \frac{\pi}{2}n$

Zeros: $2\tan\big(2(x-\pi)\big) = 0$

$$\tan\big(2(x-\pi)\big) = 0$$

$$\frac{\sin\big(2(x-\pi)\big)}{\cos\big(2(x-\pi)\big)} = 0$$

$$\sin\big(2(x-\pi)\big) = 0$$

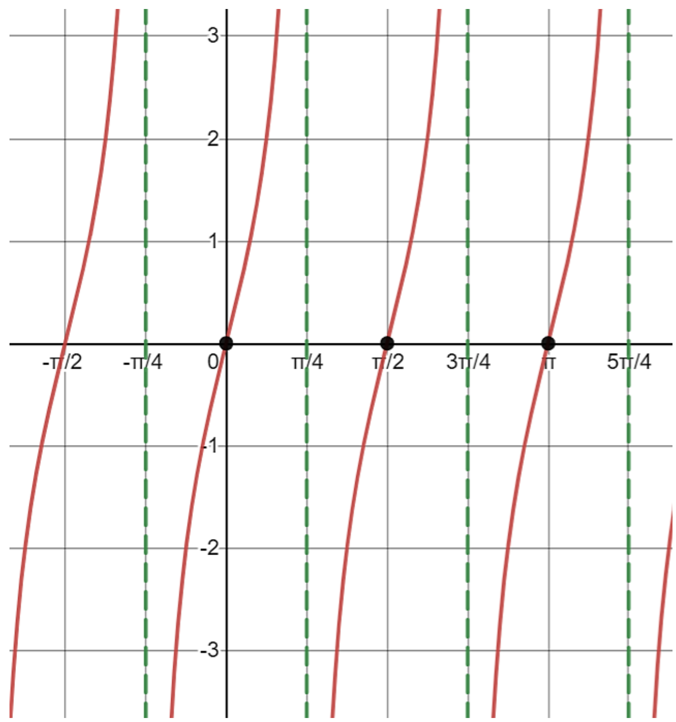
$$2(x-\pi) = 0 \ \ and \ \ 2(x-\pi) = \pi$$

$$(x-\pi) = 0 \ \ and \ \ (x-\pi) = \frac{\pi}{2}$$

$$x = \pi \ \ and \ \ x = \frac{3\pi}{2}$$

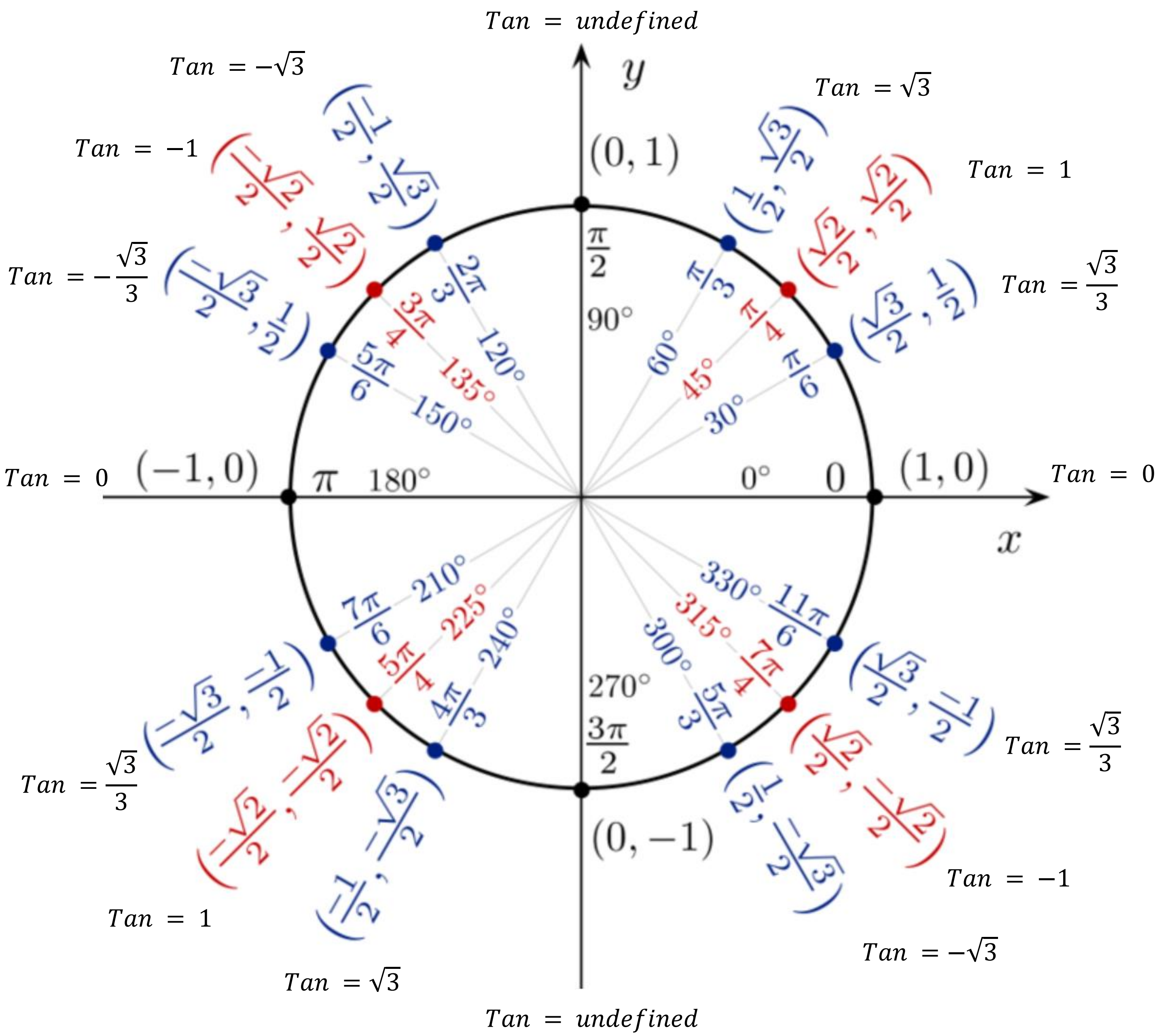
So zeros at: $x = \pi + \frac{\pi}{2}n$

$ab > 0$ so increasing



Solving Tangent Functions

We can add tangent values to the unit circle.



Just like the sine function and cosine function, we can use the unit circle to solve tangent functions, or we can solve for x using tan^{-1}, using CAST.

Ex: Solve $-3\tan\left(\frac{\pi}{3}(x+2)\right)=\sqrt{3}$

Period:

$$period=\frac{\pi}{\frac{\pi}{3}}=\pi\left(\frac{3}{\pi}\right)=3$$

$$-3\tan\left(\frac{\pi}{3}(x+2)\right)=\sqrt{3}$$

$$\tan\left(\frac{\pi}{3}(x+2)\right)=-\frac{\sqrt{3}}{3}$$

$$\left(\frac{\pi}{3}(x+2)\right)=\frac{5\pi}{6} \quad and \quad \left(\frac{\pi}{3}(x+2)\right)=\frac{11\pi}{6}$$

$$x+2=\left(\frac{5\pi}{6}\right)\left(\frac{3}{\pi}\right) \quad and \quad x+2=\left(\frac{11\pi}{6}\right)\left(\frac{3}{\pi}\right)$$

$$x+2=\left(\frac{5}{2}\right) \quad and \quad x+2=\left(\frac{11}{2}\right)$$

$$x=\frac{1}{2} \quad and \quad x=\frac{7}{2}$$

$$\therefore x=\frac{1}{2}+3n$$

Ex: Solve $3\tan\big(-4(x-6)\big)=1$ when $6\leq x\leq 7$

$$Period=\frac{\pi}{4}$$

$$3\tan\big(-4(x-6)\big)=1$$

$$\tan\big(-4(x-6)\big)=\frac{1}{3}$$

$$\tan(\theta)=\frac{1}{3}$$

$$\theta_R=0.3218$$

$$\theta_1=0.3218 \quad \theta_2=\pi+0.3218$$

$$\theta_1=0.3218 \quad \theta_2=3.4633$$

$$-4(x-6)=0.3218 \;\; and \;\; -4(x-6)=3.4633$$

$$(x-6)=-0.0805 \;\; and \;\; (x-6)=-0.8658$$

$$x=5.9195 \;\; and \; x=5.1342$$

$$\therefore x=6.7049$$

Solving Tangent Inequalities

To solve inequalities:

1. Replace inequality with =
2. Solve
3. Consult sketch for solution set

Ex: Determine the intervals over which the function $f(x) = 3\tan\left(2\left(x+\frac{\pi}{2}\right)\right) + 3$ is positive.

$$3\tan\left(2\left(x+\frac{\pi}{2}\right)\right) + 3 \geq 0$$

$$3\tan\left(2\left(x+\frac{\pi}{2}\right)\right) + 3 = 0$$

$$3\tan\left(2\left(x+\frac{\pi}{2}\right)\right) = -3$$

$$\tan\left(2\left(x+\frac{\pi}{2}\right)\right) = -1$$

$$2\left(x+\frac{\pi}{2}\right) = \frac{3\pi}{4} \quad and\ 2\left(x+\frac{\pi}{2}\right) = \frac{7\pi}{4}$$

$$\left(x+\frac{\pi}{2}\right) = \frac{3\pi}{8} \quad and \quad \left(x+\frac{\pi}{2}\right) = \frac{7\pi}{8}$$

$$x = -\frac{\pi}{8} \quad and\ x = \frac{3\pi}{8}$$

$$Period = \frac{\pi}{2}$$

Asymptotes:

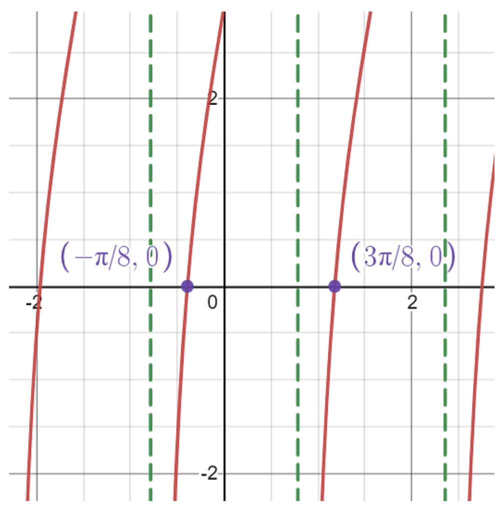
$$\cos\left(2\left(x+\frac{\pi}{2}\right)\right) = 0$$

$$2\left(x+\frac{\pi}{2}\right) = \frac{\pi}{2}\ and\ 2\left(x+\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

$$\left(x+\frac{\pi}{2}\right) = \frac{\pi}{4}\ and\ \left(x+\frac{\pi}{2}\right) = \frac{3\pi}{4}$$

$$x = -\frac{\pi}{4}\ and\ x = \frac{\pi}{4}$$

$$ab > 0\ so\ increasing$$



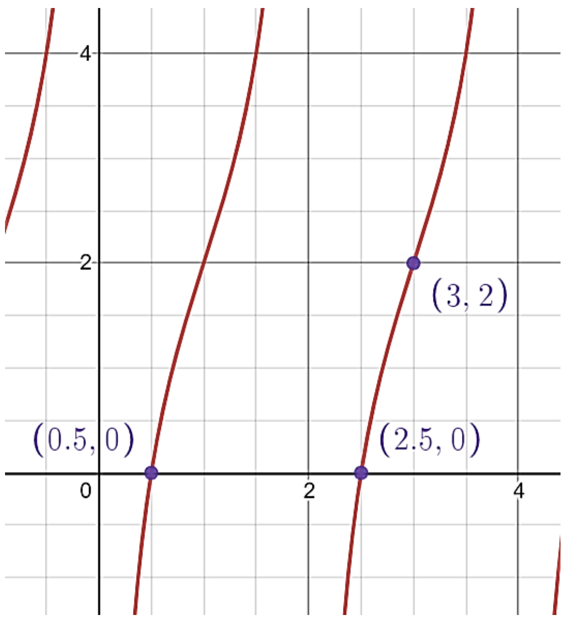
$$\therefore f(x) is\ positive\ over\ \left[\frac{3\pi}{8} + \frac{\pi}{2}n, \frac{3\pi}{4} + \frac{\pi}{2}n\right]$$

Finding the Rule of a Tangent Function

To find the rule of a tangent function:

1. Find the coordinates of the inflection point (h, k) and another point (x, y)
2. Use graph to determine period and solve for b
3. Substitute h, k, x, y, and b, then solve for a
4. Determine signs on a and b
5. Write rule

Ex: Find the rule of the tangent function graphed below.



Inflection point: (3, 2)
Additional point: (0.5, 0)

Period: $2.5 - 0.5 = 2$

$$b = \frac{\pi}{2}$$

$$f(x) = a\tan\big(b(x-h)\big) + k$$
$$0 = a\tan\left(\frac{\pi}{2}(0.5-3)\right) + 2$$
$$0 = a\tan\left(\frac{\pi}{2}(-2.5)\right) + 2$$
$$-2 = a\tan\left(\frac{-5\pi}{4}\right)$$
$$-2 = a(-1)$$
$$a = 2$$

Increasing, so $ab > 0$

$$\therefore f(x) = 2\tan\left(\frac{\pi}{2}(x-3)\right) + 2$$